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# Availability optimization in a ring-based network topology

Philippe Ezran[a,b,*], Yoram Haddad[a], Mérouane Debbah[b]

[a] Jerusalem College of Technology, Jerusalem, Israel
[b] CentraleSupélec, Gif-sur-Yvette, France

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ABSTRACT

Cellular networks are nowadays considered as a major critical infrastructure. Resiliency to failure due to disasters, weather based disruptions or malicious activities is essential. In the case of ring topology, because of delay and availability requirements, a wireless network connected to an aggregation node must sometimes be split into several rings. In this paper, we study the availability optimization in a ring-based network topology for a given number of cellular sites and a given size of rings. We prove that if each ring includes 3 nodes, the problem can be solved in a polynomial time, while for bigger rings, the problem is NP-hard. In this latter case, we provide approximation methods based on linear programming in order to converge to the solution.



## 1. Introduction

The choice of a network technology is mainly a matter of cost, availability and resiliency. For a given technology, these parameters will also impact the choice of the topology.

While fiber is capital intensive (cost function of distance) and offers limited availability, wireless is highly cost effective and flexible [1]. Besides, fiber is more expensive in urban areas. The choice of fiber in urban areas may be cost-effective for very short distances only. However, this comparison must be considered cautiously: a large part of capex expenses for fiber is civil engineering. Civil engineering costs can represent up to 80% of total cost, especially in urban areas. In many cases and particularly in developing areas, these civil engineering expenses must be done anyway for infrastructure investment (road, rail, pipeline, electricity). Therefore, the relevant parameter which has to be taken into consideration for the fiber/wireless cost comparison is the additional cost generated by the fiber.

Availability is a key parameter which quantifies network performance. This parameter is closely related to reliability. The difference between these two concepts is that reliability refers to failure-free operation during an interval, while availability refers to failure-free operation at a given instant of time [2].

Unavailability is generally defined as the sum of the unavailabilities of the network nodes [3]. According to this definition, an unavailability which impacts a great number of nodes in the network gets a higher ponderation than an unavailability which impacts a small number of nodes. However, this definition may lead to unavailabilities greater than 100%. For this reason, in this paper, we define the unavailability as the percentage of time for which all or part of the network is down.

While the availability of an optical fiber connection is all or nothing, line-of-sight and propagation considerations must be taken into account in wireless links. Automatic Coding Modulation enables a microwave link to use lower modulations in degraded conditions. This difference has an impact on the network resiliency, which is the ability of the network to provide and maintain an acceptable level of service in the face of faults and challenges to normal operations [4].

In the case of a radio network, the main parameters which come into account are propagation factors and infrastructural considerations. The most common topologies used in radio backhaul networks are trees and rings or a combination of both. Since tree topology generally offers shorter paths and lower costs, while ring topology generally ensures a better availability, a ring-tree combination can be an efficient solution to cumulate the advantages of both technologies [5].

Various causes of a network failure are identified in [4]: unusual traffic load, accidents and human mistakes, large-scale disasters, malicious attacks, environmental challenges and failures at a lower layer. A relevant network availability strategy must reduce as much as possible the failure probability of any link in the network and add redundancy in order to minimize the impact of a single link failure on the availability of the network nodes.

* Corresponding author.
*E-mail address:* ezran@g.jct.ac.il (P. Ezran).

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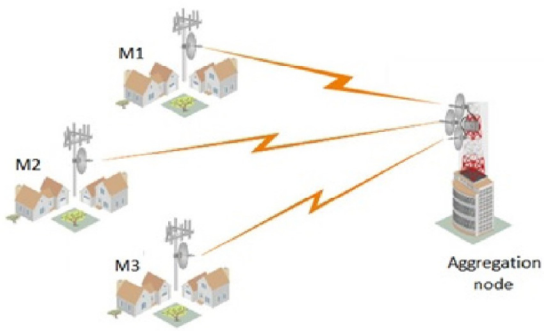


**Fig. 1.** Backhaul network. The aggregation node handles all the traffic produced by and to nodes M1, M2, M3.

Statistical approaches have been proposed in order to optimize availability [6,7] for systems subject to random failures. These approaches are based upon maintenance considerations for a partially observable system.

Backhaul can be made of fiber or microwave radio. In both cases, the goal is to connect the base stations (BS) to the core network. In some cases, when the gateway to the core is not far, this can be performed in one hop. But in rural areas or in Ultra Dense Networks, where there are a huge number of small BS to connect, this can require multiple hops. In this paper, we consider the Microwave Radio technology as the medium to perform the backhaul. We assume that we have a large number of BS to be connected to a single aggregation node which itself will be connected to the core network (Fig. 1). This latter connection is assumed to be wired and therefore out of the scope of this paper. Making a single large ring raises serious delay issues since the Backhaul for a BS might require several hops. In addition, it might raise serious availability issues since the disconnection of two links can affect a large number of BS. Therefore, it could be preferable to split the network into several rings.

In this paper, we will study the question of topology optimization from the point of view of availability maximization. Given an aggregation node and $n$ cellular sites, what is the best topology based on rings, each one of them including the aggregation node, which maximizes availability?

The paper is organized as follows: we first build a simplified model in Section 2 which provides a basic understanding regarding the relation between ring size and availability. In Section 3, we use existing results from graph theory in order to discuss the general model. Approximation methods based on linear programming are proposed in Section 4. Concluding remarks are given in Section 5.

## 2. Simplified model

In a first step, we build a simplified model, based on the following five assumptions. Though the last two assumptions of this model are not realistic, this simplified approach will enable us to draw basic conclusions regarding backhaul network topologies.

Assumptions:

- the network includes $n$ cellular sites (in addition to the aggregation node);
- the network topology is made of $k$ rings;
- for $1 \le i \le k$ ring $i$ includes $n_i$ cellular sites and the aggregation node; $n_1 \ge n_2 \ge \cdots \ge n_k \ge 2$;
- same failure probability for all links: $p$;
- failure events are uncorrelated.

$n$ and $n_i$ are related by the following equation:

$$n = \sum_{i=1}^{k} n_i \tag{1}$$

Availability: the condition for availability is that all cellular sites are connected to the aggregation node. This condition is fulfilled if there is no more than one failure in each ring.

$$A = \prod_{i=1}^{k} \left( (1-p)^{n_i+1} + (n_i+1)p(1-p)^{n_i} \right) \tag{2}$$

If $p \ll 1$, this expression can be approximated by its second-order Taylor development:

$$A = 1 - n\frac{p^2}{2} - \frac{p^2}{2}\sum_{i=1}^{k} {n_i}^2 + o(p^2) \tag{3}$$

Therefore,

$$A = 1 - n\frac{p^2}{2} - \frac{p^2}{2}\left( kV(n_i) + \frac{n^2}{k} \right) + o(p^2) \tag{4}$$

where $V(n_i)$ is the empirical variance of the $n_i$ distribution:

$$V(n_i) = \frac{1}{k}\sum_{i=1}^{k}\left(n_i - \frac{n}{k}\right)^2 = \frac{1}{k}\sum_{i=1}^{k} {n_i}^2 - \left(\frac{n}{k}\right)^2 \tag{5}$$

Therefore, increasing the number of rings reduces the maximum path length and unavailability. On the other hand, it requires more antennas. In any case, given the number of rings, it is preferable that the empirical variance of the ring size distribution be as small as possible.

For a given number of rings $k$, the maximum availability is obtained when the empirical variance is minimized, which means when the numbers of cellular sites in the rings are as close as possible to $\frac{n}{k}$. Let $q$ and $r$ be the quotient and the remainder of the Euclidean division of $n$ by $k$:

$$n = qk + r;\ 0 \le r \le k-1 \tag{6}$$

Then, $n_1 = \cdots = n_r = q+1$ and $n_{r+1} = \cdots = n_k = q$.

Therefore, the best availability is:

$$A_{k,p} = \left((1-p)^{q+2} + (q+2)p(1-p)^{q+1}\right)^r \left((1-p)^{q+1} + (q+1)p(1-p)^q\right)^{k-r} \tag{7}$$

$$A_{k,p} = 1 - n\frac{p^2}{2} - \frac{p^2}{2}\left(r(q+1)^2 + (k-r)q^2\right) + o(p^2) \tag{8}$$

$$A_{k,p} = 1 - \frac{p^2}{2}(n + kq^2 + 2rq + r) + o(p^2) \tag{9}$$

$A_{k,\,p}$ is an increasing function of $k$ and a decreasing function of $p$. Of course, since the total number of antennas is $2n + 2k$, increasing $k$ increases the cost.

The results above are illustrated with the following numerical application:

$p = 0.01$
$n = 100$
$2 \le k \le 50$

The cost and the availability are growing functions of the number of rings $k$. This defines a curve of feasible. According to the price the operator is ready to pay for a given level of availability, it is possible to define an acceptable set. Any point of the feasible curve which is inside the acceptable set is a relevant choice for the operator (Fig. 2).

It should be noted that this conclusion cannot be generalized to $n$ rings, each one of them including one cellular node: in this case,

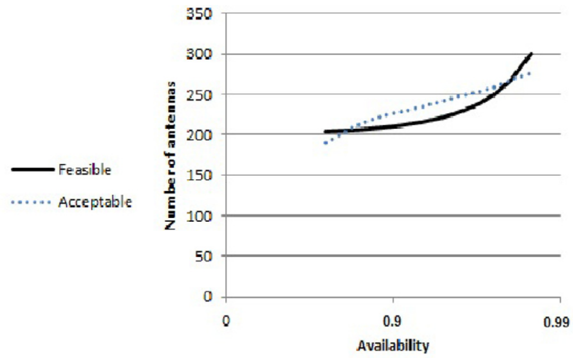


**Fig. 2.** Cost-availability balance.

the network topology would be a star topology and a cellular site would be unavailable in case of single failure. The availability of a star topology network including $n$ cellular sites is:

$$A = (1-p)^n = 1 - np + o(p) \tag{10}$$

As a consequence, availability is maximized when all the rings include 2 nodes, in addition to the aggregation node. Assuming that $n$ is even, then $k = \frac{n}{2}$ and

$$A = \left((1-p)^3 + 3p(1-p)^2\right)^{\frac{n}{2}} = (1 - 3p^2 + 2p^3)^{\frac{n}{2}}$$
$$= 1 - \frac{3}{2}np^2 + o(p^2) \tag{11}$$

## 3. General model

We now assume that links may have various failure probabilities. The network includes one aggregation node ($O$) and $n$ cellular sites $M_1$, $M_2$, ..., $M_n$.

Let $p_i$ be the failure probability of the link $OM_i$ and $p_{ij}$ the failure probability of the link $M_iM_j$.

Assuming that each ring includes the aggregation node, a ring may be defined by an ordered sequence of cellular sites.

Let us define:

$V = \{M_1, M_2, ..., M_n\}$.

$R$: the set of rings including the aggregation node and cellular sites of $V$.

A ring $r$ of $R$ can be defined with an $m$-tuple $(M_{i_1}, M_{i_2}, ..., M_{i_m})$, made from the ordered list of the nodes of $r$ starting from the aggregation node but not including it.

The availability of $r$ is:

$$A(r) = (1 - p_{i_1})(1 - p_{i_m}) \prod_{l=1}^{m-1} (1 - p_{i_l i_{l+1}})$$
$$+ p_{i_1}(1 - p_{i_m}) \prod_{l=1}^{m-1} (1 - p_{i_l i_{l+1}}) + p_{i_m}(1 - p_{i_1}) \prod_{l=1}^{m-1} (1 - p_{i_l i_{l+1}})$$
$$+ (1 - p_{i_1})(1 - p_{i_m}) \prod_{l=1}^{m-1} (1 - p_{i_l i_{l+1}}) \sum_{j=1}^{m-1} \frac{p_{i_j i_{j+1}}}{1 - p_{i_j i_{j+1}}} \tag{12}$$

For a given $s \le n$, we try to maximize the following expression:

$$\max_{\substack{r_1 \cup \ldots \cup r_s = V \\ r_i \cap r_j = \varnothing}} \prod_{i=1}^{k} A(r_i)$$

*Particular case.* $\frac{n}{2}$ rings, each one including the aggregation node and two cellular sites

In this section, we assume the following:

- $n$ is an even number;
- the network includes $\frac{n}{2}$ rings, each one including the aggregation node and 2 cellular sites;
- for $1 \le i \le n$, $p_i$ is the failure probability of the link $OM_i$;
- for $1 \le i, j \le n$, $p_{ij}$ is the failure probability of the link $M_iM_j$;
- failure events are uncorrelated.

We can calculate the availability of the ring $OM_iM_j$.

$$A_{ij} = 1 - p_ip_j - p_ip_{ij} - p_jp_{ij} + 2p_ip_jp_{ij} \tag{13}$$

We try to maximize the expression:

$$\prod_{(O,M_i,M_j)\in R} A_{ij} \tag{14}$$

which is equivalent to maximize the expression:

$$\sum_{(O,M_i,M_j)\in R} \log A_{ij} \tag{15}$$

The problem can be regarded as a search of a perfect matching in a weighted graph: given $G = (V, E, w)$ an undirected weighted graph, the goal is to compute a perfect matching (ie a subset of edges $E' \subseteq E$ such that each node in $V$ has exactly one incident edge in $E'$) for a maximum total weight $w(E')$.

The maximum 2-ring division problem can be solved efficiently (in polynomial time) as followed: given a network as described above, an undirected weighted graph $G = (V, E)$ should be constructed where $V = \{M_1, M_2, \ldots, M_n\}$ and $E = (M_i, M_j) | 1 \le i < j \le n$ (a full graph). The weight function $w : E \to \mathbb{R}$ is defined as followed: $\forall i, j, 1 \le i < j \le n, w(M_i, M_j) = \log A_{ij}$. Then, due to Gabow [8], finding a maximum 2-ring division in the original network is equivalent to finding a matching $\mathcal{M}$ in $G$ such that for each matching $\mathcal{M}'$ in $G$, $\sum_{(M_i,M_j)\in\mathcal{M}} w(M_i, M_j) \ge \sum_{(M_i,M_j)\in\mathcal{M}'} w(M_i, M_j)$.

This problem is a well-known problem called maximum weighted matching. In 1964, Jack Edmonds was the first to develop a polynomial time algorithm to solve this problem [9]. A straight forward implementation of Edmonds' algorithm will have a running time complexity of $O(|V|^2|E|)$, and hence in our problem $O(|V|^4)$ (because the constructed graph is fully meshed. i.e. $E = \Theta(|V|^2)$). Over the years, several variants, implementations and improvements of Edmonds' idea where suggested, some of them in [8,10,11]. Overall, the best know algorithm for a full graph has a running time complexity of $O(|V|^3)$ [10–12].

Now, solving the maximum 2-ring division problem is done in 2 phases:

1. Computing $logA_{ij}$ for each $i, j$, $1 \le i < j \le n$ and constructing an undirected weighted full graph $G$, as described earlier.
2. Solving the weighted maximum matching problem on $G$.

The running time complexity of phase 1 is $\binom{n}{2}\Theta(1) + \Theta(n^2) = \Theta(n^2)$. The running time complexity of phase 2 is $O(n^3)$. Therefore, the running time complexity of the proposed algorithm for solving the maximum 2-ring division problem is $O(n^3)$. Hence, the decision problem corresponding to the maximum 2-ring division problem is in P.

Conclusion: it is possible to connect an even number $n$ of cellular sites with $n/2$ rings, each of one including 2 cellular sites and the aggregation node. The running time is $O(n^3)$.

*General case.* $\frac{n}{k}$ rings, each one including the aggregation node and $k$ cellular sites; $k \ge 3$

In this section, we assume the following:

- $n$ is a multiple of $k$;
- the network includes $\frac{n}{k}$ rings, each one including the aggregation node and $k$ cellular sites;

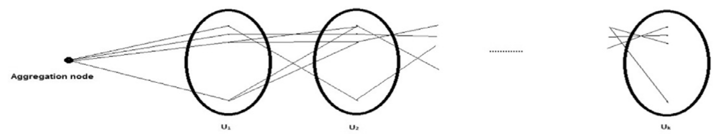


**Fig. 3.** Particular case of $k$-ring division.

- for $1 \le i \le n$, $p_i$ is the failure probability of the link $OM_i$;
- for $1 \le i, j \le n$, $p_{ij}$ is the failure probability of the link $M_iM_j$;
- failure events are uncorrelated.

At first, we investigate the relation between the general maximum $k$-ring division problem and an NP-Complete problem.

Let $\mathcal{P}_k(n)$ be the set of $k$-combinations of $\{1, 2, \ldots, n\}$. Given a family of sets $F \subseteq \mathcal{P}_k(n)$ for $k \ge 3$, a $k$-set packing of $\{1, 2, \ldots, n\}$ is a set $S \subseteq F$ such that $\forall s_1, s_2 \in S, s_1 \cap s_2 = \varnothing$. The maximum $k$-set packing problem (MSP) is to find a $k$-set packing $S$ of $\{1, 2, \ldots, n\}$ such that for each $k$-set packing $S'$ of $\{1, 2, \ldots, n\}$, $|S| \ge |S'|$. The corresponding decision problem $(d - MSP)$ is a well-known NP-Complete problem [13,14].

We define the maximum production [0, 1) weighted $k$-set packing (MPWSP) as followed: given a family $F = \mathcal{P}_k(n)$ where $n = mk$ for some $m \in \mathbb{N}$ and a weight function $w : F \to [0, 1)$, the MPWSP problem is to find a $k$-set packing $S$ of $\{1, 2, \ldots, n\}$ such that for each $k$-set packing $S'$ of $F$, $\prod_{u \in S} w(u) \ge \prod_{u \in S'} w(u)$.

Let $d - MPWSP$ denote the corresponding decision problem to MPWSP. $d - MSP$ is a particular case of $d - MPWSP$ with:

- $w(X) = 1$ for $X \in F$
- $w(X) = 0$ for $X \notin F$

Therefore, $d - MPWSP$ is as least as hard as $d - MSP$. Thus, d-MPWSP is NP-hard (and in fact, d-MPWSP is NP-Complete).

Given an algorithm to solve the MPWSP problem, it can be used to solve the general maximum $k$-ring division problem as followed: let $A^{max}_{i_1 i_2 \ldots i_k}$ be the highest availability of all the rings including the $k$ nodes $i_1$, $i_2$, ..., $i_k$ and the aggregation node:

$$A^{max}_{i_1 i_2 \ldots i_k} = max(A_{j_1 j_2 \ldots j_k} | j_1 j_2 \ldots j_k \text{ is a permutation of } i_1 i_2 \ldots i_k) \tag{16}$$

and

$$c_{i_1 i_2 \ldots i_k} = log(A^{max}_{i_1 i_2 \ldots i_k}) \tag{17}$$

An instance of MPWSP could be constructed by defining a family of sets $F = \mathcal{P}_k(n)$ and a weight function $w(i_1, i_2, i_k) = A^{max}_{i_1 i_2 \ldots i_k}$. Clearly, a solution to the constructed MPWSP instance yields a solution to the original maximum $k$-ring division problem.

Reciprocally, let us consider the following particular case:

$$V = U_1 \cup \ldots \cup U_k$$

$$|U_1| = \ldots = |U_k| = \frac{n}{k}$$

$$\forall u \in U_1,\ p_u = 0$$

$$\forall u \in U_2 \cup \ldots \cup U_k,\ p_u = 1$$

$$\forall u_i \in U_i, \forall u_j \in U_j, |j - i| \neq 1 \to p_{ij} = 1$$

$$\forall u_i \in U_i, \forall u_j \in U_j, |j - i| = 1 \to p_{ij} \in [0, 1]$$

In this particular case, the aggregation node is connected to all the nodes of $U_1$ and no other node. Every connection between the aggregation node and anyone of the nodes of $U_1$ is assumed to be free of failure-risk. A node in a given subset $U_i$ can be connected only to the nodes belonging to the adjacent sets $U_{i-1}$ and $U_{i+1}$. (Fig. 3).

Then, the maximization of $c_{i_1 i_2 \ldots i_k}$ is a $k$-dimensional matching problem, which is known to be NP hard [15] for $k \ge 3$. Therefore, the general problem is NP hard for $k \ge 3$.

## 4. Approximation methods

Since the general maximum k-ring division problem is NP-hard for $k \ge 3$, we propose hereafter approximation methods in order to converge to the solution.

### 4.1. Formalization as an integer linear programming problem

We can present the k-ring division problem as an ILP: the idea is to define binary variables which correspond to a $k$-ring.

$$P = \max \sum_{\{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k(n)} c_{i_1 i_2 \ldots i_k} x_{i_1 i_2 \ldots i_k} \tag{18}$$

subject to

$$\sum_{\substack{\{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k(n) \\ j \in \{i_1, i_2, \ldots, i_k\}}} x_{i_1 i_2 \ldots i_k} = 1, \forall j \in \{1, 2, \ldots, n\} \tag{19}$$

$$x_{i_1 i_2 \ldots i_k} \in \{0, 1\}, \forall \{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k(n) \tag{20}$$

The purpose of this method is to characterize the network topology by binary values: $x_{i_1 i_2 \ldots i_k} = 1$ if the nodes $i_1, i_2, \ldots, i_k$, together with the aggregation node, form a ring, and $x_{i_1 i_2 \ldots i_k} = 0$ else.

Constraints (19) and (20) forces each node $j$ to be in exactly one k-ring.

General ILP is known to be NP-hard [12]. However, linear programming can be solved in polynomial time. By replacing constraint (20) in (18) with the constraint:

$$x_{i_1 i_2 \ldots i_k} \ge 0, \forall \{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k(n) \tag{21}$$

(also known as LP relaxation) we get a polynomial-time solvable linear program.

Without loss of generality, we can assume that $c_{i_1 i_2 \ldots i_k} > 0$ for each $\{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k(n)$, since we can always add any constant to all the coefficients $c_{i_1 i_2 \ldots i_k}$. Doing that does not change the set of vectors that maximizes the problem, because due to the constraints, each feasible vector contains exactly $\frac{n}{k}$ ones and $\binom{n}{k} - \frac{n}{k}$ zeros. Therefore, adding $K$ to all the coefficients $c_{i_1 i_2 \ldots i_k}$ is equivalent to adding the constant $K\frac{n}{k}$ to the original objective function.

**Lemma 4.1.** *Assuming $c_{i_1 i_2 \ldots i_k} > 0$ for each $\{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k(n)$, a vector $x$ which maximizes the system*

$$P' = \max \sum_{\{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k(n)} c_{i_1 i_2 \ldots i_k} x_{i_1 i_2 \ldots i_k} \tag{22}$$

*subject to*

$$\sum_{\substack{\{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k(n) \\ j \in \{i_1, i_2, \ldots, i_k\}}} x_{i_1 i_2 \ldots i_k} \le 1, \forall j \in \{1, 2, \ldots, n\} \tag{23}$$

$$x_{i_1 i_2 \ldots i_k} \in \{0, 1\}, \forall \{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k(n) \tag{24}$$

*is feasible to (18).*

**Proof.** All we need to show is that $\sum_{\{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k(n) j \in \{i_1, i_2, \ldots, i_k\}} x_{i_1 i_2 \ldots i_k} = 1, \forall j \in \{1, 2, \ldots, n\}$. Assume by contradiction that there is a $j$ for which $\sum_{\{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k(n) j \in \{i_1, i_2, \ldots, i_k\}} x_{i_1 i_2 \ldots i_k} = 0$ (there is no other possibility since $x$ satisfies constraint (24); this means that there is a node $j$ that is not in any $k$-ring). Since $n$ is a multiplier of

$k$, there are $k-1$ other nodes that are not in any $k$-ring, therefore, a new ring can be added to the sum contradicting the fact that $x$ maximizes $P'$. □

**Lemma 4.2.** *Assuming $c_{i_1 i_2 \ldots i_k} > 0$ for each $\{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k(n)$, a vector x which maximizes (22) maximizes (18).*

**Proof.** Straight from Lemma 4.1 and from the fact that any feasible vector in (18) is a feasible vector in (22). □

However, not each solution to the relaxation yields a solution to the original problem. Consider the following example:

$$n = 6; k = 3 \quad (25)$$

$$c_{124} = c_{135} = c_{236} = c_{456} = 1; \text{all other } c_{ijl} = 0 \quad (26)$$

Then, $\max \Sigma c_{ijl} x_{ijl}$ subject to (19), (21) is obtained only for:

$$x_{124} = x_{135} = x_{236} = x_{456} = \frac{1}{2}; \text{all other } x_{ijl} = 0 \quad (27)$$

*4.2. Power method*

In order to favour the emergence of a maximum which has exclusively integer coordinates, we introduce an exponent $\alpha$. The purpose of this exponent is to penalize potential non-integer solutions.

For each $\alpha > 0$ we define:

$$P_\alpha = \max \sum_{\{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k(n)} c_{i_1 i_2 \ldots i_k} x^{\alpha}_{i_1 i_2 \ldots i_k} \quad (28)$$

subject to

$$\sum_{\substack{\{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k(n) \\ j \in \{i_1, i_2, \ldots, i_k\}}} x_{i_1 i_2 \ldots i_k} \le 1, \forall j \in \{1, 2, \ldots, n\} \quad (29)$$

$$x_{i_1 i_2 \ldots i_k} \ge 0, \forall \{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k(n) \quad (30)$$

We also define $x_\alpha$ a vector in $\mathbb{R}^{\binom{n}{k}}$ which maximizes (28) (can be one of many if there is more than one vector which maximizes $P_\alpha$).

**Lemma 4.3.** *If $x_\alpha \in \{0, 1\}^{\binom{n}{k}}$, then $x_\alpha$ maximizes (22).*

**Proof.** $x_\alpha$ is obviously feasible to (22). Assume by contradiction that there is $y$ which satisfies (23) and (24) for which $P'(y) > P'(x_\alpha)$. Since all coordinates of $x_\alpha$ and $y$ are 0 or 1, $P'(x_\alpha) = P_\alpha(x_\alpha)$ and $P'(y) = P_\alpha(y)$. Therefore, $P_\alpha(y) > P_\alpha(x_\alpha)$, contradicting the fact that $x_\alpha$ maximizes (28). □

**Lemma 4.4.** $\lim_{\alpha\to\infty} P_\alpha(x_\alpha) = \lim_{\alpha\to\infty} P_\alpha(\lfloor x_\alpha \rfloor)$

**Proof.**

$$\lim_{\alpha\to\infty} P_\alpha(x_\alpha) = \lim_{\alpha\to\infty} \sum_{x_{i_1 i_2 \ldots i_k} \text{coordinates of } x_\alpha} c_{i_1 i_2 \ldots i_k} x^{\alpha}_{i_1 i_2 \ldots i_k}$$

$$= \lim_{\alpha\to\infty} \sum_{x_{i_1 i_2 \ldots i_k} \text{coordinates of } x_\alpha} c_{i_1 i_2 \ldots i_k} \lfloor x_{i_1 i_2 \ldots i_k} \rfloor^{\alpha} = \lim_{\alpha\to\infty} P_\alpha(\lfloor x_\alpha \rfloor) \quad (31)$$

The second equality derives from the fact that constraints (29) and (30) force that $0 \le x_{i_1 i_2 \ldots i_k} \le 1$. Hence:

- if $x_{i_1 i_2 \ldots i_k} = 1$, then $c_{i_1 i_2 \ldots i_k} x^{\alpha}_{i_1 i_2 \ldots i_k} = c_{i_1 i_2 \ldots i_k} \lfloor x_{i_1 i_2 \ldots i_k} \rfloor^{\alpha} = c_{i_1 i_2 \ldots i_k}$
- if $x_{i_1 i_2 \ldots i_k} < 1$, then $\lim_{\alpha\to\infty} c_{i_1 i_2 \ldots i_k} x_{i_1 i_2 \ldots i_k}{}^{\alpha} = 0$ and $c_{i_1 i_2 \ldots i_k} \lfloor x_{i_1 i_2 \ldots i_k} \rfloor^{\alpha} = 0$ □

**Theorem 4.5.** *Assuming $c_{i_1 i_2 \ldots i_k} > 0$ for each $\{i_1, i_2, \ldots, i_k\} \in \mathcal{P}_k(n)$, as $\alpha \to \infty$, $\lfloor x_\alpha \rfloor$ maximizes (18).*

**Proof.** From Lemma 4.4, we conclude that $\lfloor x_\alpha \rfloor$ maximizes (28). Then, from Lemma 4.3, we conclude that $\lfloor x_\alpha \rfloor$ maximizes (22). Finally, from Lemma 4.2, we conclude that $\lfloor x_\alpha \rfloor$ maximizes (18). □

Theorem 4.5 offers an alternative way to solve (18) for a large $\alpha$ and select the floor values of the elements in $x_\alpha$.

## 5. Conclusion

Availability is maximized when the number of rings is high and the ring size distribution is regular. In this paper, we show that the partition of a network including an aggregation node and $n$ cellular sites into $\frac{n}{2}$ rings, each one including the aggregation node and 2 cellular sites, can be solved in a time of $O(n^3)$. Regarding a partition with larger rings, the problem is similar to a $k$-set partition problem, which is NP-hard for $k \ge 3$. We propose an approximation method, based on linear programming and use of an exponent aimed to accelerate the convergence.

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**Philippe Ezran** graduated from the Ecole Polytechnique, France, in 1987 and from Télécom ParisTech, France, in 1989. He worked for France Télécom R&D, Paris, France, from 1989 to 2000 and for Crédit Agricole, Paris, France, from 2000 to 2010. In 2011–2012, he worked for Brightsource Industries, Jerusalem, Israel. He is presently working at the Jerusalem College of Technology on network optimization within the framework of Software Defined Networking.