

Enhanced Max-Min SINR for Uplink Cell-Free Massive MIMO Systems

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Abstract—In this paper, we consider the max-min signal-to-interference plus noise ratio (SINR) problem for the uplink transmission of a cell-free Massive multiple-input multiple-output (MIMO) system. Assuming that the central processing unit (CPU) and the users exploit only the knowledge of the channel statistics, we first derive a closed-form expression for uplink rate. In particular, we enhance (or maximize) user fairness by solving the max-min optimization problem for user rate, by power allocation and choice of receiver coefficients, where the minimum uplink rate of the users is maximized with available transmit power at the particular user. Based on the derived closed-form expression for the uplink rate, we formulate the original user max-min problem to design the optimal receiver coefficients and user power allocations. However, this max-min SINR problem is not jointly convex in terms of design variables and therefore we decompose this original problem into two sub-problems, namely, receiver coefficient design and user power allocation. By iteratively solving these sub-problems, we develop an iterative algorithm to obtain the optimal receiver coefficient and user power allocations. In particular, the receiver coefficients design for a fixed user power allocation is formulated as generalized eigenvalue problem whereas a geometric programming (GP) approach is utilized to solve the power allocation problem for a given set of receiver coefficients. The performance of the proposed max-min SINR algorithm is evaluated in terms of the achieved user uplink rate. Numerical results confirm a three-fold increase in system rate over existing schemes in the literature.

Keywords: Cell-free Massive MIMO, convex optimization, max-min SINR problem, geometric programming, generalized eigenvalue problem.

I. INTRODUCTION

Massive multiple-input multiple-output (MIMO) is one of the most promising techniques for 5th Generation (5G) networks due to its potential for significant rate enhancement and spectral as well as energy efficiency [1], [2]. This Massive MIMO technique can be implemented through three alternative ways [3]: i) single cell Massive MIMO, in which a large number of users are simultaneously served by a single base station (BS) with a much larger number of antennas; ii) multi cell Massive MIMO, where the simulation area is divided into several cells with one BS in each cell where the BSs jointly establish communications with distributed users in the cells; and iii) cell-free Massive MIMO: there is no cell

concept and the simulation area is not divided into cells. In cell-free Massive MIMO, randomly distributed access points (APs) jointly serve distributed users [3].

In this paper, we propose a max-min signal-to-interference plus noise ratio (SINR) approach for an uplink cell-free Massive MIMO system. In [4], the authors investigate the problem of max-min SINR in a single-cell Massive MIMO system. A similar max-min SINR problem referred to SINR balancing has been considered for cognitive radio network in [5]–[7]. In [8], the same max-min SINR problem is considered through appropriate user power allocation where a bisection search method is utilized to determine the optimal solution. However, a novel approach to significantly improve all users' performance is proposed in this paper by designing optimal receiver coefficients and user power allocation. By employing maximal ratio combining (MRC) at the receiver, we first derive a closed-form expression for the average uplink rate of the users. Based on these user power allocations and receiver coefficients, we formulate the corresponding max-min SINR problem, which is not jointly convex in terms of the design parameters. In order to realize a solution for this non-convex problem, we decompose the original problem into two sub-problems: receiver coefficient design and user power allocation. An iterative algorithm is proposed that successively solves these two sub-problems while one of the design variables (i.e., user power allocation or receiver coefficients) is fixed. The receiver coefficient design is formulated into a generalized eigenvalue problem [9] whereas a geometric programming (GP) approach [10] is exploited to solve the user power allocation problem. The performance of the proposed scheme in terms of the user rate is significantly higher than that of the scheme proposed in [8].

The contributions and the results of our work are summarized as follows: **1)** For the considered cell-free Massive MIMO system, we derive the average user rate in the uplink. **2)** Based on the derived user rate, we propose a novel max-min SINR approach to significantly improve the SINR performance in terms of the achieved user rate. The original max-min problem formulation is not convex and therefore we decompose the original problem into two sub-problems and propose an iterative algorithm to yield the optimal solution.

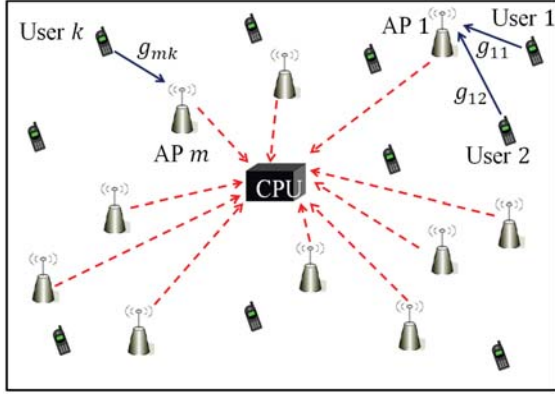


Figure 1. The uplink of a cell-free Massive MIMO system with K users and M APs. The dashed lines denote the uplink channels and the solid lines present the backhaul links from the APs to the central processing unit (CPU).

3) The user power allocation and the receiver coefficient design sub-problems are solved through the GP approach and the generalized eigenvalue problem, respectively. 4) Finally, we present complexity analysis of the proposed scheme. 5) Numerical results are provided to validate the superiority of the proposed algorithm in comparison with the scheme proposed in [8].

Outline: The remainder of the paper is organized as follows. Section II describes the system model. Section III investigates the corresponding performance analysis. The proposed max-min SINR approach is presented in Section IV. Numerical results and complexity analysis are provided in Section V and Section VI, respectively. Finally, Section VII concludes the paper.

II. SYSTEM MODEL

We consider uplink transmission in a cell-free Massive MIMO system with M randomly distributed single-antenna APs and K randomly distributed single-antenna users in the area, as shown in Fig. 1. The channel coefficient between the k th user and the m th AP, g_{mk} , is modeled as [8]

$$g_{mk} = \sqrt{\beta_{mk}} h_{mk}, \quad (1)$$

where β_{mk} denotes the large-scale fading and $h_{mk} \sim \mathcal{CN}(0, 1)$ represents small-scale fading between the k th user and the m th AP.

A. Uplink Channel Estimation

In order to estimate the channel coefficients in the uplink, the APs employ an minimum mean square error (MMSE) estimator. All pilot sequences used in the channel estimation phase are collected in a matrix $\Phi \in \mathbb{C}^{\tau \times K}$, where τ is the length of the pilot sequence for each user and the k th column, ϕ_k , represents the pilot sequence used for the k th user. After performing a de-spreading operation, the MMSE estimate of the channel coefficient between the k th user and the m th AP is given by [8]

$$\hat{g}_{mk} = c_{mk} \left(\sqrt{\tau p_p} g_{mk} + \sqrt{\tau p_p} \sum_{k' \neq k} g_{mk'} \phi_k^H \phi_{k'} + \phi_k^H \mathbf{n}_{p,m} \right), \quad (2)$$

where each element of $\mathbf{n}_{p,m}$, $n_{p,m} \sim \mathcal{CN}(0, 1)$, denotes the noise at the m th antenna, p_p represents the normalized signal-to-noise ratio (SNR) of each pilot sequence (which we define in Section V), and c_{mk} is given by

$$c_{mk} = \frac{\sqrt{\tau p_p} \beta_{mk}}{\tau p_p \sum_{k'=1}^K \beta_{mk'} |\phi_k^H \phi_{k'}|^2 + 1}. \quad (3)$$

The estimated channels in (2) are used by the APs to design the receiver coefficients and determine power allocations at users to maintain user fairness. In this paper, we investigate the cases of both random pilot assignment and orthogonal pilots in cell-free Massive MIMO. Here the term "orthogonal pilots" refers to the case where unique orthogonal pilots are assigned to all users, while in "random pilot assignment" each user is randomly assigned a pilot sequence from a set of orthogonal sequences of length τ ($< K$), following the approach of [8], [11].

B. Uplink Data Transmission

In this subsection, we consider the uplink data transmission, where all users send their signals to the APs. The transmitted signal from the k th user is represented by

$$x_k = \sqrt{q_k} s_k, \quad (4)$$

where s_k ($\mathbb{E}\{|s_k|^2\} = 1$) and q_k denote respectively the transmitted symbol and the transmit power at the k th user. The received signal at the m th AP from all users is given by

$$y_m = \sqrt{\rho} \sum_{k=1}^K g_{mk} \sqrt{q_k} s_k + n_m, \quad (5)$$

where $n_m \sim \mathcal{CN}(0, 1)$ is the noise at the m th AP. In addition, MRC is employed at the APs. More precisely, the received signal at the m th AP, y_m , is first multiplied with \hat{g}_{mk}^* . The resulting $\hat{g}_{mk}^* y_m$ is then forwarded to the CPU for signal detection. In order to improve achievable rate, the forwarded signal is further multiplied by a receiver filter coefficient at the CPU. The aggregated received signal at the CPU can be written as

$$\begin{aligned} r_k &= \sum_{m=1}^M u_{mk} \hat{g}_{mk}^* y_m \\ &= \sqrt{\rho} \sum_{k'=1}^K \sum_{m=1}^M u_{mk} \hat{g}_{mk}^* g_{mk'} \sqrt{q_{k'}} s_{k'} + \sum_{m=1}^M u_{mk} \hat{g}_{mk}^* n_m. \end{aligned} \quad (6)$$

By collecting all the coefficients u_{mk} , $\forall m$ corresponding to the k th user, we define $\mathbf{u}_k = [u_{1k}, u_{2k}, \dots, u_{Mk}]^T$ and without loss of generality, it is assumed that $\|\mathbf{u}_k\| = 1$. The optimal solution of \mathbf{u}_k , q_k , $\forall k$ for the considered max-min SINR approach is investigated in Section IV.

III. PERFORMANCE ANALYSIS

In this section, we derive the average user rate for the considered system model in the previous section by following a similar approach to that in [8]. Note that the main difference between the proposed approach and the scheme in [8] is the new set of receiver coefficients which are introduced at

the CPU to improve the achievable user rates. The benefits of the proposed approach in terms of achieved user uplink rate is demonstrated through numerical simulation results in Section V. In deriving the achievable rates of each user, it is assumed that the CPU exploits only the knowledge of channel statistics between the users and APs in detecting data from the received signal in (7). Without loss of generality, the aggregated received signal in (7) can be written as

$$\begin{aligned}
r_k &= \sqrt{\rho} \mathbb{E} \left\{ \underbrace{\sum_{m=1}^M u_{mk} \hat{g}_{mk}^* g_{mk} \sqrt{q_k}}_{\text{DS}_k} \right\} s_k \quad (7) \\
&+ \sqrt{\rho} \left(\underbrace{\sum_{m=1}^M u_{mk} \hat{g}_{mk}^* g_{mk} \sqrt{q_k} - \mathbb{E} \left\{ \sum_{m=1}^M u_{mk} \hat{g}_{mk}^* g_{mk} \sqrt{q_k} \right\}}_{\text{BU}_k} \right) s_k \\
&+ \underbrace{\sum_{k' \neq k}^K \sqrt{\rho} \sum_{m=1}^M u_{mk} \hat{g}_{mk}^* g_{mk'} \sqrt{q_{k'}}}_{\text{IUI}_{kk'}} s_{k'} + \underbrace{\sum_{m=1}^M u_{mk} \hat{g}_{mk}^* n_m}_{\text{TN}_k},
\end{aligned}$$

where DS_k and BU_k denote the desired signal (DS) and beamforming uncertainty (BU) for the k th user, respectively, and IUI_k represents the inter-user-interference (IUI) caused by the k' th user. In addition, TN_k accounts for the total noise (TN) following the MRC detection. The average SINR of the received signal in (7) can be defined by considering the worst-case of the uncorrelated Gaussian noise as follows [8]:

$$\text{SINR}_k^{\text{UP}} = \frac{|\text{DS}_k|^2}{\mathbb{E}\{|\text{BU}_k|^2\} + \sum_{k' \neq k}^K \mathbb{E}\{|\text{IUI}_{kk'}|^2\} + \mathbb{E}\{|\text{TN}_k|^2\}}. \quad (8)$$

Based on the SINR definition in (8), the achievable uplink rate of the k th user is defined in the following theorem.

Theorem 1. *By employing MRC detection at APs, the achievable uplink rate of the k th user in the Cell-free Massive MIMO system with K randomly distributed single-antenna users and M single-antenna APs is given by (9) (defined at the beginning of the next page).*

Note that in (9), we have

$$\mathbf{\Gamma}_k = [\gamma_{1k}, \gamma_{2k}, \dots, \gamma_{Mk}]^T, \quad (10a)$$

$$\mathbf{u}_k = [u_{1k}, u_{2k}, \dots, u_{Mk}]^T, \quad (10b)$$

$$\mathbf{\Delta}_{kk'} = \left[\frac{\gamma_{1k} \beta_{1k'}}{\beta_{1k}}, \frac{\gamma_{2k} \beta_{2k'}}{\beta_{2k}}, \dots, \frac{\gamma_{Mk} \beta_{Mk'}}{\beta_{Mk}} \right]^T, \quad (10c)$$

$$\mathbf{R}_k = \text{diag} [\gamma_{1k}, \gamma_{2k}, \dots, \gamma_{Mk}], \quad (10d)$$

$$\mathbf{D}_{kk'} = \text{diag} [\beta_{1k'} \gamma_{1k}, \beta_{2k'} \gamma_{2k}, \dots, \beta_{Mk'} \gamma_{Mk}], \quad (10e)$$

Proof: Please refer to Appendix A.

IV. PROPOSED MAX-MIN SINR SCHEME

In this section, we formulate the max-min SINR problem in cell-free massive MIMO, where the minimum uplink user rate between users is maximized while satisfying the transmit

Algorithm 1 Proposed Algorithm to Solve P_1

1. Initialize $\mathbf{q}^{(0)} = [q_1^{(0)}, q_2^{(0)}, \dots, q_K^{(0)}]$, $i = 1$
 2. Repeat
 3. $i = i + 1$
 4. Set $\mathbf{q}^{(i)} = \mathbf{q}^{(i-1)}$ and find the optimal receiver coefficients $\mathbf{U}^{(i)} = [\mathbf{u}_1^{(i)}, \mathbf{u}_2^{(i)}, \dots, \mathbf{u}_K^{(i)}]$ through solving the generalized eigenvalue Problem P_2 in (12)
 5. Compute $\mathbf{q}^{(i+1)}$ through solving Problem P_4 in (14).
 6. Go back to Step 3 and repeat until required accuracy.
-

power constraint at each user. This max-min rate problem can be formulated as follows:

$$\begin{aligned}
P_1 : \max_{\mathbf{q}_k, \mathbf{u}_k} \quad & \min_{k=1, \dots, K} R_k \quad (11) \\
\text{s.t.} \quad & \|\mathbf{u}_k\| = 1, \quad \forall k, \quad 0 \leq q_k \leq p_{\max}^{(k)}, \quad \forall k,
\end{aligned}$$

where $p_{\max}^{(k)}$ is the maximum transmit power available at user k . Problem P_1 is not jointly convex in terms of \mathbf{u}_k and power allocation q_k , $\forall k$. Therefore, this problem cannot be directly solved through existing convex optimization software. To tackle this non-convexity issue, we divide the original Problem P_1 into two sub-problems: receiver coefficient design (i.e. \mathbf{u}_k) and the power allocation problem. To obtain a solution for Problem P_1 , these sub-problems are alternately solved as explained in the following subsections.

A. Receiver Coefficients Design

In this subsection, we solve the receiver coefficient design problem to maximize the uplink rate of each user for a given set of transmit power allocation at all users. These coefficients (i.e., \mathbf{u}_k , $\forall k$) can be obtained by interdependently maximizing the uplink SINR of each user. Hence, the optimal coefficients for all users for a given set of transmit power allocation can be determined by solving the following optimization problem:

$$\begin{aligned}
P_2 : \quad & \max_{\mathbf{u}_k} \frac{\mathbf{u}_k^H (q_k \mathbf{\Gamma}_k \mathbf{\Gamma}_k^H) \mathbf{u}_k}{\mathbf{u}_k^H \left(\sum_{k' \neq k}^K q_{k'} |\boldsymbol{\phi}_k^H \boldsymbol{\phi}_{k'}|^2 \mathbf{\Delta}_{kk'} \mathbf{\Delta}_{kk'}^H + \sum_{k'=1}^K q_{k'} \mathbf{D}_{kk'} + \frac{1}{\rho} \mathbf{R}_k \right) \mathbf{u}_k}, \\
\text{s.t.} \quad & \|\mathbf{u}_k\| = 1, \quad \forall k. \quad (12)
\end{aligned}$$

Problem P_2 is a generalized eigenvalue problem [9], where the optimal solutions can be obtained by determining the generalized eigenvalue of the matrix pair $\mathbf{A}_k = q_k \mathbf{\Gamma}_k \mathbf{\Gamma}_k^H$ and $\mathbf{B}_k = \sum_{k' \neq k}^K q_{k'} |\boldsymbol{\phi}_k^H \boldsymbol{\phi}_{k'}|^2 \mathbf{\Delta}_{kk'} \mathbf{\Delta}_{kk'}^H + \sum_{k'=1}^K q_{k'} \mathbf{D}_{kk'} + \frac{1}{\rho} \mathbf{R}_k$ corresponding to the maximum generalized eigenvalue.

B. Power Allocation

In this subsection, we solve the power allocation problem for a set of fixed receiver coefficients. The power allocation problem can be formulated into the following max-min problem:

$$P_3 : \max_{q_k} \quad \min_{k=1, \dots, K} \text{SINR}_k \quad (13)$$

$$R_k = \log_2 \left(1 + \frac{\mathbf{u}_k^H \left(q_k \mathbf{\Gamma}_k \mathbf{\Gamma}_k^H \right) \mathbf{u}_k}{\mathbf{u}_k^H \left(\sum_{k' \neq k}^K q_{k'} |\phi_k^H \phi_{k'}|^2 \Delta_{kk'} \Delta_{kk'}^H + \sum_{k'=1}^K q_{k'} \mathbf{D}_{kk'} + \frac{1}{\rho} \mathbf{R}_k \right) \mathbf{u}_k} \right). \quad (9)$$

Table I
COMPUTATIONAL COMPLEXITY OF DIFFERENT PROBLEMS

Problems	Required arithmetic operations
Problem P_2 , given by (12)	$\mathcal{O}(KM^3)$
Problem P_4 , given by (14)	$\mathcal{O}(K^{\frac{7}{2}})$
The scheme in [8]	$\log_2(\frac{t_{\max} - t_{\min}}{\epsilon}) \mathcal{O}(K^4)$

$$\text{s.t. } 0 \leq q_k \leq p_{\max}^{(k)}, \quad \forall k,$$

Without loss of generality, Problem P_3 can be rewritten by introducing a new slack variable as

$$P_4 : \quad \max_{t, q_k} t \quad (14)$$

$$\text{s.t. } 0 \leq q_k \leq p_{\max}^{(k)}, \quad \forall k, \quad \text{SINR}_k \geq t, \quad \forall k.$$

Proposition 1: Problem P_4 can be formulated into a GP.

Proof: Please refer to Appendix B.

Therefore, this problem can be efficiently solved through existing convex optimization software. Based on these two sub-problems, an iterative algorithm is developed by alternately solving each sub-problem in each iteration. The proposed algorithm is summarized in Algorithm 1.

V. CONVERGENCE ANALYSIS

In this section, the convergence analysis of the proposed Algorithm 1 is provided. Two sub-problems are alternately solved to determine the solution to Problem P_1 . At each iteration, one of the design parameters is determined by solving the corresponding sub-problem while other design variable is fixed. Note that each sub-problem provides an optimal solution for the other given design variable. At the i th iteration, the receiver filter coefficients $\mathbf{u}_k^{(i)}$, $\forall k$ are determined for a given power allocation $\mathbf{q}^{(i)}$ and similarly, the power allocation $\mathbf{q}^{(i+1)}$ is updated for a given set of receiver filter coefficients $\mathbf{u}_k^{(i)}$, $\forall k$. The optimal power allocation $\mathbf{q}^{(i+1)}$ obtained for a given $\mathbf{u}_k^{(i)}$ achieves an uplink rate greater than or equal to that of the previous iteration. In addition, the power allocation $\mathbf{q}^{(i)}$ is also a feasible solution in determining $\mathbf{q}^{(i+1)}$ as the receiver filter coefficients $\mathbf{u}_k^{(i+1)}$, $\forall k$ are determined for a given $\mathbf{q}^{(i)}$. This reveals that the achieved uplink rate monotonically increases with each iteration, which can be also observed from the simulation results presented in Fig. 4. As the achievable uplink max-min rate is upper

bounded by a certain value for a given set of per-user power constraints, the proposed algorithm converges to a particular solution. Fortunately, the proposed Algorithm 1 converges to the optimal solution, as we will prove by establishing the uplink-downlink duality in the following section.

VI. COMPLEXITY ANALYSIS

Here, we provide the computational complexity analysis for the proposed Algorithm 1, which solves a generalized eigenvalue problem P_2 and a GP (convex optimization problem) P_4 at each iteration. For the receiver filter coefficient design in P_2 , given by (12), an eigenvalue solver requires approximately $\mathcal{O}(KM^3)$ flops [12], [13]. In addition, a standard GP in Problem P_4 , defined in (14), can be solved with complexity equivalent to $\mathcal{O}(K^{\frac{7}{2}})$ [14, Chapter 10]. However, for the scheme in [8], the iterative bisection search method in [8] solves a Second order cone programming (SOCP) at each iteration. The complexity of SOCP is $\mathcal{O}(K^4)$ in each iteration [15], [16]. Note that the total number of iterations to solve a bisection search method in [8] is given by $\log_2(\frac{t_{\max} - t_{\min}}{\epsilon})$, where ϵ refers to a predetermined threshold [10]. The number of arithmetic operations required for Algorithm 1 are provided in Table I.

VII. NUMERICAL RESULTS AND DISCUSSION

In this section, we provide numerical simulation results to validate the performance of the proposed max-min SINR approach with different parameters. A cell-free Massive MIMO system with M APs and K single antenna users is considered in a $D \times D$ simulation area, where both APs and users are randomly distributed. In the following subsections, we define the simulation parameters and then present the corresponding simulation results.

A. Simulation Parameters

To model the channel coefficients between users and APs, the coefficient β_{mk} is given by $\beta_{mk} = \text{PL}_{mk} \cdot 10^{\frac{\sigma_{sh} z_{mk}}{10}}$ where PL_{mk} is the path loss from the k th user to the m th AP, and $10^{\frac{\sigma_{sh} z_{mk}}{10}}$ denotes the shadow fading with standard deviation σ_{sh} , and $z_{mk} \sim \mathcal{N}(0, 1)$ [8]. The noise power is given by $P_n = \text{BW} k_B T_0 W$, where $\text{BW} = 20$ MHz denotes the bandwidth, $k_B = 1.381 \times 10^{-23}$ represents the Boltzmann constant, and $T_0 = 290$ (Kelvin) denotes the noise temperature. Moreover, $W = 9$ dB, and denotes the noise figure [8]. It is assumed that that \bar{P}_p and $\bar{\rho}$ denote the pilot sequence and the uplink data, respectively, where $P_p = \frac{\bar{P}_p}{P_n}$ and $\rho = \frac{\bar{\rho}}{P_n}$. In simulations, we set $\bar{P}_p = 100$ mW and $\bar{\rho} = 100$ mW. Similar to [8], we suppose the simulation area is wrapped around at the edges which can simulate an

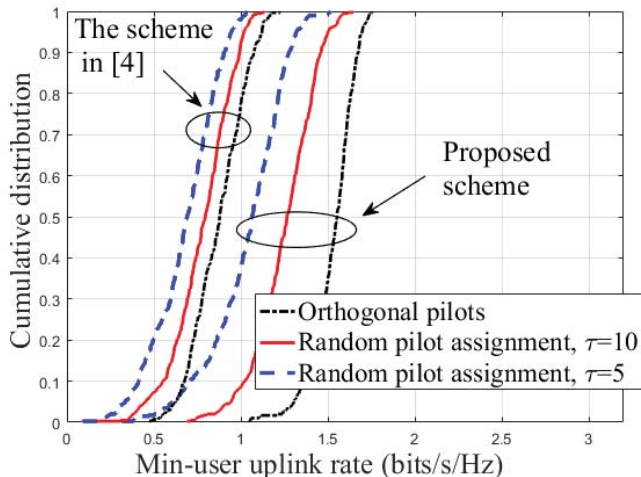


Figure 2. The cumulative distribution of the per-user uplink rate, with orthogonal and random pilots for $M = 60$, $K = 20$ and $D = 1$ km.

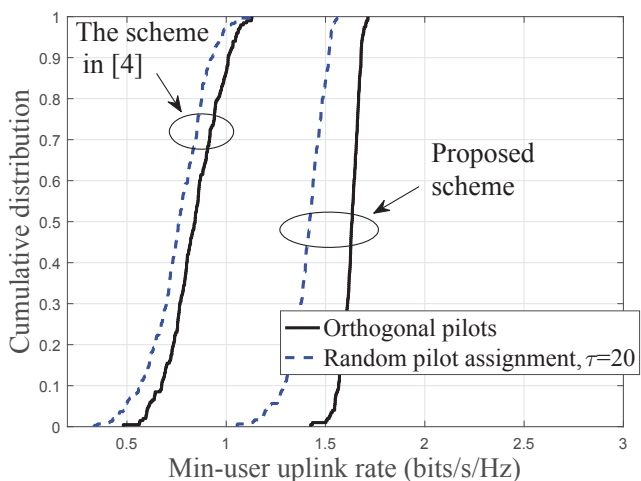


Figure 3. The cumulative distribution of the per-user uplink rate, with random pilots for $M = 100$, $K = 40$, $\tau = 20$, and $D = 1$ km.

area without boundaries. Hence, the square simulation area has eight neighbors. We evaluate the rate of the system over 300 random realizations of the locations of APs, users and shadowing.

B. Simulation Results

In this subsection, we investigate the effect of the max-min SINR problem on the system performance. Fig. 2 compares the cumulative distribution of the achievable uplink rates for our proposed algorithm with the power allocation scheme in [8], for three cases of orthogonal pilots, random pilots with $\tau = 10$ and $\tau = 5$ for the length of pilot sequence. In Fig. 2, $M = 60$ APs and $K = 20$ users are randomly distributed through the simulation area of size 1×1 km². As the figure shows, the performance of the proposed scheme is almost three times than that of the scheme in [8]. In Fig. 3, we compare the performance of the proposed max-min SINR approach with the scheme in [8] for the case

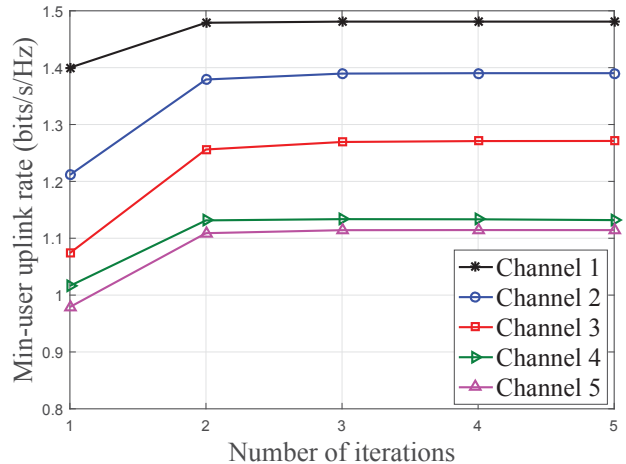


Figure 4. The convergence of the proposed max-min SINR approach for $M = 60$, $K = 20$, $\tau = 10$, and $D = 1$ km.

of $M = 100$ APs, $K = 40$ users and $\tau = 20$ as the length of the pilot sequence. Fig. 3 shows the superiority of the proposed iterative algorithm over the power allocation scheme in [8]. Moreover, Fig. 3 demonstrates that the rate of the proposed max-min SINR approach is more concentrated around the median. Fig. 4 investigates the convergence of the proposed max-min SINR algorithm for a set of different channel realizations. The figure shows that the proposed algorithm converges after a few iterations, while the minimum rate of the users increases with the iteration number.

VIII. CONCLUSIONS

We have considered the max-min optimization problem in cell-free Massive MIMO systems, and propose an efficient solution that maximizes the smallest of the uplink rate of the users. We propose to divide the original max-min problem into two sub-problems which can be iteratively solved by exploiting generalized eigenvalue problem and GP. The simulation results showcased the effectiveness of the proposed scheme in terms of maximising the smallest of the uplink rate of the users compared with existing schemes.

APPENDIX A

PROOF OF THEOREM 1

The desired signal for the user k is given by

$$\begin{aligned} DS_k &= \mathbb{E} \left\{ \sum_{m=1}^M u_{mk} \hat{g}_{mk}^* g_{mk} \sqrt{q_k} \right\} \\ &= \sqrt{q_k} \sum_{m=1}^M u_{mk} \gamma_{mk}. \end{aligned} \quad (15)$$

Hence,

$$|DS_k|^2 = q_k \left(\sum_{m=1}^M u_{mk} \gamma_{mk} \right)^2. \quad (16)$$

Moreover, the term $\mathbb{E}\{|\text{BU}_k|^2\}$ can be obtained as

$$\begin{aligned} \mathbb{E}\{|\text{BU}_k|^2\} &= \rho \mathbb{E}\left\{\left|\sum_{m=1}^M u_{mk} \hat{g}_{mk}^* g_{mk} \sqrt{q_k}\right.\right. \\ &\quad \left.\left. - \rho \mathbb{E}\left\{\sum_{m=1}^M u_{mk} \hat{g}_{mk}^* g_{mk} \sqrt{q_k}\right\}\right|^2\right\} \\ &= \rho \sum_{m=1}^M q_k u_{mk}^2 \left(\mathbb{E}\left\{|\hat{g}_{mk}^* g_{mk} - \mathbb{E}\{\hat{g}_{mk}^* g_{mk}\}|^2\right\}\right) \\ &= \rho q_k \sum_{m=1}^M u_{mk}^2 \gamma_{mk} \beta_{mk}, \end{aligned} \quad (17)$$

where the last equality comes from the analysis in [8, Appendix A], Appendix A) and using the following fact; $\gamma_{mk} = \mathbb{E}\{|\hat{g}_{mk}|^2\} = \sqrt{\tau p_p} \beta_{mk} c_{mk}$. The term $\mathbb{E}\{|\text{IU}_{kk'}|^2\}$ is obtained as

$$\begin{aligned} \mathbb{E}\{|\text{IU}_{kk'}|^2\} &= \rho \mathbb{E}\left\{\left|\sum_{m=1}^M u_{mk} \hat{g}_{mk}^* g_{mk'} \sqrt{q_{k'}}\right|^2\right\} \\ &= \rho \mathbb{E}\left\{\left|\sum_{m=1}^M c_{mk} u_{mk} g_{mk'} \sqrt{q_{k'}}\right.\right. \\ &\quad \left.\left.\left(\sqrt{\tau p_p} \sum_{i=1}^K g_{mi} \phi_k^H \phi_i + \phi_k^H \mathbf{n}_{p,m}\right)^*\right|^2\right\} \\ &= \rho q_{k'} \mathbb{E}\left\{\underbrace{\left|\sum_{m=1}^M c_{mk} u_{mk} g_{mk'} \tilde{n}_{mk}^*\right|^2}_A\right\} \\ &\quad + \rho \tau p_p \mathbb{E}\left\{\underbrace{q_{k'} \left|\sum_{m=1}^M c_{mk} u_{mk} g_{mk'} \left(\sum_{i=1}^K g_{mi} \phi_k^H \phi_i\right)^*\right|^2}_B\right\}, \end{aligned} \quad (18)$$

where the third equality in (18) is due to the fact that for two independent random variables X and Y and $\mathbb{E}\{X\} = 0$, we have $\mathbb{E}\{|X+Y|^2\} = \mathbb{E}\{|X|^2\} + \mathbb{E}\{|Y|^2\}$ [8, Appendix A]. Since $\tilde{n}_{mk} = \phi_k^H \mathbf{n}_{p,m} \sim \mathcal{CN}(0, 1)$ is independent from the term $g_{mk'}$ similar to [8, Appendix A], the term A in (18) immediately is given by $A = q_{k'} \sum_{m=1}^M c_{mk}^2 u_{mk}^2 \beta_{mk'}$. The term B in (18) can be obtained as

$$\begin{aligned} B &= \tau p_p q_{k'} \mathbb{E}\left\{\underbrace{\left|\sum_{m=1}^M c_{mk} u_{mk} |g_{mk'}|^2 \phi_k^H \phi_{k'}\right|^2}_C\right\} \\ &\quad + \tau p_p q_{k'} \mathbb{E}\left\{\underbrace{\left|\sum_{m=1}^M c_{mk} u_{mk} g_{mk'} \left(\sum_{i \neq k'}^K g_{mi} \phi_k^H \phi_i\right)^*\right|^2}_D\right\}. \end{aligned} \quad (19)$$

The first term in (19) is given by

$$\begin{aligned} C &= \tau p_p q_{k'} \mathbb{E}\left\{\left|\sum_{m=1}^M c_{mk} u_{mk} |g_{mk'}|^2 \phi_k^H \phi_{k'}\right|^2\right\} \\ &= 2\tau p_p q_{k'} |\phi_k^H \phi_{k'}|^2 \sum_{m=1}^M c_{mk}^2 u_{mk}^2 \beta_{mk'}^2 + \tau p_p q_{k'} \\ &\quad \mathbb{E}\left\{\left|\phi_k^H \phi_{k'}\right|^2 \sum_{m=1}^M \sum_{n \neq m}^M c_{mk} c_{nk} u_{mk} u_{nk} |g_{mk'}|^2 |g_{nk'}|^2\right\} \\ &= \tau p_p q_{k'} |\phi_k^H \phi_{k'}|^2 \sum_{m=1}^M c_{mk}^2 u_{mk}^2 \beta_{mk'}^2 \\ &\quad + q_{k'} |\phi_k^H \phi_{k'}|^2 \left(\sum_{m=1}^M u_{mk} \gamma_{mk} \frac{\beta_{mk'}}{\beta_{mk}}\right)^2, \end{aligned} \quad (20)$$

where the last equality is derived based on the fact $\gamma_{mk} = \sqrt{\tau p_p} \beta_{mk} c_{mk}$. The second term in (19) can be obtained as

$$\begin{aligned} D &= \tau p_p q_{k'} \mathbb{E}\left\{\left|\sum_{m=1}^M c_{mk} u_{mk} g_{mk'} \left(\sum_{i \neq k'}^K g_{mi} \phi_k^H \phi_i\right)^*\right|^2\right\} \\ &= \tau p_p \sum_{m=1}^M \sum_{i \neq k'}^K q_{k'} c_{mk}^2 u_{mk}^2 \beta_{mk'} \beta_{mi} |\phi_k^H \phi_i|^2. \end{aligned} \quad (21)$$

Hence, (18) can be written as

$$\begin{aligned} \mathbb{E}\{|\text{IU}_{kk'}|^2\} &= q_{k'} \underbrace{\sum_{m=1}^M c_{mk}^2 u_{mk}^2 \beta_{mk'}}_{C_1} \\ &\quad + \tau p_p q_{k'} |\phi_k^H \phi_{k'}|^2 \sum_{m=1}^M c_{mk}^2 u_{mk}^2 \beta_{mk'}^2 \\ &\quad + \tau p_p q_{k'} \sum_{m=1}^M \sum_{i \neq k'}^K c_{mk}^2 u_{mk}^2 \beta_{mk'} \beta_{mi} |\phi_k^H \phi_i|^2 \\ &\quad + q_{k'} |\phi_k^H \phi_{k'}|^2 \left(\sum_{m=1}^M u_{mk} \gamma_{mk} \frac{\beta_{mk'}}{\beta_{mk}}\right)^2, \end{aligned} \quad (22)$$

and

$$\begin{aligned} C_2 &= \tau p_p q_{k'} |\phi_k^H \phi_{k'}|^2 \sum_{m=1}^M c_{mk}^2 u_{mk}^2 \beta_{mk'}^2 \\ &\quad + \tau p_p q_{k'} \underbrace{\sum_{m=1}^M \sum_{i \neq k'}^M c_{mk}^2 u_{mk}^2 \beta_{mk'} \beta_{mi} |\phi_k^H \phi_i|^2}_{C_3}. \end{aligned} \quad (24)$$

For the last term of (24), we have

$$\begin{aligned}
C_3 &= \tau p_p q_{k'} \sum_{m=1}^M \sum_{i \neq k'}^K c_{mk}^2 u_{mk}^2 \beta_{mk'} \beta_{mi} |\phi_k^H \phi_i|^2 \\
&= \tau p_p q_{k'} \left(\sum_{m=1}^M u_{mk}^2 c_{mk} \beta_{mk'} \sum_{i=1}^K c_{mk} \beta_{mi} |\phi_k^H \phi_i|^2 \right. \\
&\quad \left. - \sum_{m=1}^M u_{mk}^2 c_{mk}^2 \beta_{mk'} |\phi_k^H \phi_{k'}|^2 \right) \\
&= \sqrt{\tau p_p} q_{k'} \sum_{m=1}^M u_{mk}^2 c_{mk} \beta_{mk'} \beta_{mk} - q_{k'} \sum_{m=1}^M u_{mk}^2 c_{mk}^2 \beta_{mk'} \\
&\quad - \tau p_p q_{k'} \sum_{m=1}^M u_{mk}^2 c_{mk}^2 \beta_{mk'} |\phi_k^H \phi_{k'}|^2, \tag{25}
\end{aligned}$$

where in the last step, we used equation (3). As a result, $C_1 + C_2 = \sqrt{\tau p_p} q_{k'} \sum_{m=1}^M u_{mk}^2 c_{mk} \beta_{mk'} \beta_{mk}$. Then finally we have

$$\begin{aligned}
\mathbb{E} \left\{ |\mathbf{I} \mathbf{U}_{kk'}|^2 \right\} &= \rho q_{k'} \left(\sum_{m=1}^M u_{mk}^2 \beta_{mk'} \gamma_{mk} \right) \\
&\quad + \rho q_{k'} |\phi_k^H \phi_{k'}|^2 \left(\sum_{m=1}^M u_{mk} \gamma_{mk} \frac{\beta_{mk'}}{\beta_{mk}} \right)^2. \tag{26}
\end{aligned}$$

The total noise for the user k is given by

$$\mathbb{E} \left\{ |\mathbf{T} \mathbf{N}_k|^2 \right\} = \mathbb{E} \left\{ \left| \sum_{m=1}^M u_{mk} \hat{g}_{mk}^* n_m \right|^2 \right\} = \sum_{m=1}^M u_{mk}^2 \gamma_{mk}, \tag{27}$$

where the last equality is due to the fact that the terms \hat{g}_{mk} and n_m are uncorrelated. Finally, by substituting (16), (17), (27) and (27) into (8), SINR of k th user is obtained by (9), which completes the proof of Theorem 1. ■

APPENDIX B

PROOF OF PROPOSITION 1

The standard form of GP is defined as follows [10]:

$$P_5 : \min \quad f_0(\mathbf{x}) \tag{28}$$

$$\begin{aligned}
\text{s.t.} \quad & f_i(\mathbf{x}) \leq 1, \quad i = 1, \dots, m, \\
& g_i(\mathbf{x}) = 1, \quad i = 1, \dots, p,
\end{aligned}$$

where f_0 and f_i are posynomial and g_i are monomial functions. Moreover, $\mathbf{x} = \{x_1, \dots, x_n\}$ represent the optimization variables. The SINR constraint in (14) is not a posynomial function in its form, however it can be rewritten into the following posynomial function:

$$\frac{\mathbf{u}_k^H \left(\sum_{k' \neq k}^K q_{k'} |\phi_k^H \phi_{k'}|^2 \Delta_{kk'} \Delta_{kk'}^H + \sum_{k'=1}^K q_{k'} \mathbf{D}_{kk'} + \frac{1}{p} \mathbf{R}_k \right) \mathbf{u}_k}{\mathbf{u}_k^H (q_k \Gamma_k \Gamma_k^H) \mathbf{u}_k} < \frac{1}{t}, \quad \forall k. \tag{29}$$

By applying a simple transformation, (29) is equivalent to the following inequality:

$$q_k^{-1} \left(\sum_{k' \neq k}^K a_{kk'} q_{k'} + \sum_{k'=1}^K b_{kk'} q_{k'} + c_k \right) < \frac{1}{t}, \tag{30}$$

where $a_{kk'} = \frac{\mathbf{u}_k^H (|\phi_k^H \phi_{k'}|^2 \Delta_{kk'} \Delta_{kk'}^H) \mathbf{u}_k}{\mathbf{u}_k^H (\Gamma_k \Gamma_k^H) \mathbf{u}_k}$, $b_{kk'} = \frac{\mathbf{u}_k^H \mathbf{D}_{kk'} \mathbf{u}_k}{\mathbf{u}_k^H (\Gamma_k \Gamma_k^H) \mathbf{u}_k}$, and finally $c_k = \frac{\mathbf{u}_k^H \mathbf{R}_k \mathbf{u}_k}{p \mathbf{u}_k^H (\Gamma_k \Gamma_k^H) \mathbf{u}_k}$. The transformation in (30) shows that the left-hand side of (29) is a polynomial function. Therefore, the power allocation Problem P_4 is a standard GP (convex problem), where the objective function and constraints are monomial and polynomials, which completes the proof of Proposition 1. ■

REFERENCES

- [1] A. Zappone, L. Sanguinetti, G. Bacci, E. A. Jorswieck, and M. Debbah, "Energy-efficient power control: a look at 5G wireless technologies," *IEEE Trans. Signal Process.*, vol. 64, no. 7, pp. 1668–1683, Apr. 2016.
- [2] E. Björnson, E. A. Jorswieck, M. Debbah, and B. Ottersten, "Multiobjective signal processing optimization: the way to balance conflicting metrics in 5G systems," *IEEE Signal Process. Mag.*, vol. 31, no. 6, pp. 14–23, Oct. 2014.
- [3] H. Yang and T. L. Marzetta, "Capacity performance of multicell large-scale antenna systems," in *Proc. IEEE Allerton Conference*, Oct. 2013, pp. 668–675.
- [4] H. Sifaou, A. Kammoun, L. Sanguinetti, M. Debbah, and M. Alouini, "Max–min SINR in large-scale single-cell MU-MIMO: asymptotic analysis and low-complexity transceivers," *IEEE Trans. Signal Process.*, vol. 65, no. 7, pp. 1841–1854, Apr. 2017.
- [5] K. Cumanan, R. Krishna, L. Musavian, and S. Lambotharan, "Joint beamforming and user maximization techniques for techniques for cognitive radio networks based on branch and bound method," *IEEE Trans. Wireless Commun.*, vol. 9, no. 10, pp. 3082–3092, Oct. 2010.
- [6] K. Cumanan, L. Musavian, S. Lambotharan, and A. B. Gershman, "SINR balancing technique for downlink beamforming in cognitive radio networks," *IEEE Signal Process. Lett.*, vol. 17, no. 2, pp. 133–136, Feb. 2010.
- [7] K. Cumanan, Y. Rahulamathavan, S. Lambotharan, and Z. Ding, "MMSE based beamforming techniques for relay broadcast channel," *IEEE Trans. Veh. Technol.*, vol. 62, no. 8, pp. 4045–4051, Oct. 2013.
- [8] H. Q. Ngo, A. Ashikhmin, H. Yang, E. G. Larsson, and T. L. Marzetta, "Cell-free Massive MIMO versus small cells," *IEEE Trans. Wireless Commun.*, vol. 16, no. 3, pp. 1834–1850, Mar. 2017.
- [9] G. Golub and C. V. Loan, *Matrix Computations*, 2nd ed. Baltimore, MD: The Johns Hopkins Univ. Press, 1996.
- [10] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, UK: Cambridge University Press, 2004.
- [11] H. Ahmadi, A. Farhang, N. Marchetti, and A. MacKenzie, "A game theoretic approach for pilot contamination avoidance in Massive MIMO," *IEEE Wireless Commun. Lett.*, vol. 5, no. 1, pp. 12–15, Feb. 2016.
- [12] C. Chien, H. Su, and H. Li, "Joint beamforming and power allocation for MIMO relay broadcast channel with individual SINR constraints," *IEEE Trans. Veh. Technol.*, vol. 63, no. 4, pp. 1660–1677, 2004.
- [13] G. H. Golub and C. F. V. Loan, *Matrix Computations*, 3rd ed. Baltimore, MD, USA: Johns Hopkins Univ. Press, 1996.
- [14] Y. Nesterov and A. Nemirovsky, *Interior-point polynomial methods in convex programming*. Studies in Applied Mathematics, SIAM, Philadelphia, 1994.
- [15] M. S. Lobo, L. Vandenberghe, S. Boyd, and H. Lebret, "Applications of second-order cone programming," *Linear Algebra Appl.*, pp. 193–228, Nov. 1998.
- [16] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optim. Meth. Softw.*, vol. 11, p. 625–653, Aug. 1999.