# A Bayesian Game-Theoretic Approach for Distributed Resource Allocation in Fading Multiple Access Channels

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**Abstract**

A Bayesian game-theoretic model is developed to design and analyze the resource allocation problem in $K$-user fading multiple access channels (MAC), where users are assumed to selfishly maximize their average achievable rates with incomplete information about the fading channel gains. In such a game-theoretic study, the central question is whether a Bayesian equilibrium exists, and if so, whether the network operates efficiently at the equilibrium point. We prove that there exists exactly one Bayesian equilibrium in our game. Furthermore, we study the network sum-rate maximization problem by assuming that users coordinate to the symmetric strategy profile. This result also serves as an upper bound for the Bayesian equilibrium. Finally, simulation results are provided to show the network efficiency at the unique Bayesian equilibrium, and compare it with other strategies.

## I. Introduction

Fading multiple access channel (MAC) is a basic wireless channel model that allows several transmitters connected to the same receiver to transmit over it and share its capacity. The capacity region of fading MAC and the optimal resource allocation algorithms have been characterized and well studied in many pioneering works with different information assumptions [1]-[4]. However, in order to achieve the full capacity region, it usually requires a central computing resource (a scheduler with comprehensive knowledge of the network information) to globally

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allocate the system resources. This process is centralized, it involves feedback and overhead communication whose load scales linearly with the number of transmitters and receivers in the network. In addition, with the fast evolution of wireless techniques, this centralized network infrastructure begins to expose its weakness in many aspects, e.g., slow reconfiguration against varying environment, increased computational complexity, etc. This is especially crucial for femto-cell networks where it is quite difficult to centralize the information due to a limited capacity backhaul. Moreover, the high density of base stations would increase the cost of centralizing the information.

In recent years, increased research interest has been given to self-organizing wireless networks in which mobile devices allocate resource in a decentralized manner [5]. Tools from game theory [6] have been widely applied to study the resource allocation and power control problems in fading MAC [7], as well as many other types of networks, such as orthogonal frequency division multiplexing (OFDM) [8], multiple input and multiple output (MIMO) channels [9], [10], and interference channels [11]. Typically, the game-theoretic models used in these previous works assume that the information/knowledge about other devices is available to all devices. However, this assumption is hardly met in practice. In practical wireless communication scenarios, mobile devices can have local information but can barely access to global information on the network status.

A static non-cooperative game has been introduced in the context of two-user fading MAC, known as “waterfilling game” [7]. By assuming that users compete with transmission rates as utility and transmit powers as moves, the authors show that there exists a unique Nash equilibrium [12] which corresponds to the maximum sum-rate point of the capacity region. This claim is somewhat surprising, since in general Nash equilibrium is inefficient comparing to the Pareto optimality. However, their results rely on the fact that both transmitters have the complete knowledge of the channel state information (CSI), and in particular, perfect CSI of all transmitters in the network. As we previously pointed out, this assumption is rarely possible in practice.

Thus, this power allocation game needs to be reconstructed with some realistic assumptions made on the knowledge level of mobile devices. Under this consideration, it is of great interest to investigate scenarios in which devices have “incomplete information” about their components, e.g., a device is ware of its own channel gain, but unaware of the channel gains of other devices.

In game theory, a strategic game with incomplete information is called a “Bayesian game”. Over the last ten years, Bayesian game-theoretic tools have been used to design distributed resource allocation strategies only in a few contexts, e.g., CDMA networks [14], [15], multi-carrier interference networks [16]. The motivation of this paper is therefore to investigate how Bayesian games can be applied to study the resource allocation problems in fading MAC.

The paper is organized in the following form: In Section II, we introduce the system model and assumptions. In Section III, the $K$-user MAC is formulated as a static Bayesian game. In Section IV, we characterize the Bayesian equilibrium set. In Section V, we give a special discussion on the optimal symmetric strategy. Some numerical results are provided to show the efficiency of Bayesian equilibrium in Section VI. Finally, we close with some concluding remarks in Section VII.

## II. SYSTEM MODEL AND ASSUMPTIONS

### A. System model

We consider a time-slotted flat-fading MAC in a single-cell network, in which $K$ users are simultaneously sending information to one base station. At time $t$, the signal received by the base station can be mathematically expressed as

$$y\left(t\right)=\sum_{k=1}^{K}\sqrt{g_k\left(t\right)}x_k\left(t\right)+z\left(t\right)$$

where $x_k\left(t\right)$ and $g_k\left(t\right)$ are the input signal and fading channel gain of user $k$, $z\left(t\right)$ is a zero-mean white Gaussian noise with variance $\sigma^2$. The input signal $x_k\left(t\right)$ can be further written as

$$x_k\left(t\right)=\sqrt{p_k\left(t\right)}s_k\left(t\right)$$

where $p_k\left(t\right)$ and $s_k\left(t\right)$ are the transmitted power and data of user $k$ at time $t$.

We assume that the channel gains $g_1,\ldots,g_K$ are deterministic constants during the period of each transmission block (which is assumed to be larger than a time slot interval). Therefore, within each time slot $t$, this is simply a Gaussian multi-user channel [13]. Now, instead of considering the whole capacity region, we are interested in the single-user achievable rate (assuming that the base station uses low complexity single-user decoder [13]), i.e.,

$$R_k=\log\left(1+\frac{g_kp_k}{\sigma^2+\sum_{j=1,j\neq k}^{K}g_jp_j}\right)\qquad(1)$$

### B. Assumption of finite channel states

Before introducing our game model, we need to clarify a prior assumption.

**Assumption II.1.** *We assume that each user's channel gain $g_k$ is i.i.d. from two discrete values: $g_-$ and $g_+$ with probability $\rho_-$ and $\rho_+$, respectively. WloG, we assume $g_- < g_+$.*

On the one hand, our assumption is closely related to the way how feedback information are signalled to the transmitters. In order to get the channel information $g_k$ at the transmitter side, it requires the base station to estimate it and then feedback to user $k$ *at a given precision*. Since in digital communications, any information is represented by a finite number of bits (e.g., $x$ bits), channels gains are mapped into a set that contains a finite number of states ($2^x$ states).

On the other hand, this is a necessary assumption for analytical tractability, since in principle the functional strategic form of a player can be quite complex with both actions and states are continuous (or infinite). To avoid this problem, in [16] the authors successfully modelled a multi-carrier Gaussian interference channel as a Bayesian game with discrete (or finite) actions and continuous states. Inspired from [16], in this paper, we model the fading MAC as a Bayesian game under the assumption of continuous actions and discrete states.

## III. Game Formulation

Here, we model the $K$-user fading MAC as a Bayesian game, in which users do not have complete information. Imagine in a $K$-user MAC, the "complete information" means that, at time $t$, the channel gain realizations $g_1(t), \ldots, g_K(t)$ are known at all the transmitters $\mathrm{Tx}_1, \ldots, \mathrm{Tx}_K$; whereas the "incomplete information" means that each transmitter $\mathrm{Tx}_k$ only knows its own channel gain realization $g_k(t)$, but does not know the channel gains of other transmitters $g_{-k}(t)$.

In such a communication system, the natural object of each user is to maximize its *average* achievable rate, i.e.,

$$\begin{aligned} \max_{p_k} \quad & \mathbb{E}_{\mathbf{g}}\left[\log\left(1+\frac{g_k p_k(g_k)}{\sigma^2+\sum_{j\neq k} g_j p_j(g_j)}\right)\right] \\ s.t. \quad & \mathbb{E}_{g_k}\left[p_k\left(g_k\right)\right] \leq P_k^{\max} \\ & p_k\left(g_k\right) \geq 0 \end{aligned} \tag{2}$$

where $p_k(\cdot)$ is user $k$'s transmit power strategy, $\mathbf{g} = \{g_1, \ldots, g_K\}$ is a set of channel gains, $P_k^{\max}$ is the average power constraint for user $k$. Note that under the assumption that each user has only incomplete information about the fading channel gains, user $k$'s power strategy $p_k(\cdot)$ is defined as a function of its own channel gain $g_k$.

For a given strategy set $p_{-k} = \{p_1, \dots, p_{k-1}, p_{k+1}, \dots, p_K\}$, the single-user maximization problem (2) is a convex optimization problem [17]. Via Lagrangian duality, the solution is given by the following equation

$$\mathbb{E}_{g_{-k}}\left[\frac{g_k}{\sigma^2 + g_k p_k(g_k) + \sum_{j\neq k} g_j p_j(g_j)}\right] = \lambda_k \tag{3}$$

where $g_{-k} = \{g_1, \dots, g_{k-1}, g_{k+1}, \dots, g_K\}$ and the dual variable $\lambda_k$ is chosen such that the power constraint in (2) is satisfied with equality. However, the solution of (3) depends on $p_{-k}(\cdot)$ which user $k$ does not know, and same for other users. Thus, in order to obtain the optimal power allocation, each user must adjusts its power level based on the guess of all other users' strategies. Given the following game model, each user is able to adjust its strategy according to the belief it has on the strategy of the other user.

The $K$-player MAC Bayesian game can be completely characterized as:

$$\mathcal{G}_{MAC} \triangleq \langle \mathcal{K}, \mathcal{T}, \mathcal{P}, \mathcal{Q}, \mathcal{U} \rangle$$

- Player set: $\mathcal{K} = \{1, \dots, K\}$.
- Type set: $\mathcal{T} = \mathcal{T}_1 \times \dots \times \mathcal{T}_K$ ('$\times$' stands for the Cartesian product) where $\mathcal{T}_k = \{g_-, g_+\}$, a player's type is defined as its channel gain, i.e., $g_k \in \mathcal{T}_k$.
- Action set: $\mathcal{P} = \mathcal{P}_1 \times \dots \times \mathcal{P}_K$ where $\mathcal{P}_k = [0, P_k^{\max}]$, a player's action is defined as its transmit power, i.e., $p_k \in \mathcal{P}_k$.
- Probability set: $\mathcal{Q} = \mathcal{Q}_1 \times \dots \times \mathcal{Q}_K$ where $\mathcal{Q}_k = \{\rho_-, \rho_+\}$, we have $\rho_+ = \Pr(g_k = g_+)$ and $\rho_- = \Pr(g_k = g_-)$.
- Payoff function set: $\mathcal{U} = \{u_1, \dots, u_K\}$ where $u_k$ is chosen as player $k$'s achievable rate, as defined in (1)

$$u_k(p_1, \dots, p_K) = \log\left(1 + \frac{g_k p_k(g_k)}{\sigma^2 + \sum_{j=1, j\neq k}^{K} g_j p_j(g_j)}\right) \tag{4}$$

In games of incomplete information, a player's type represents any kind of private information that is relevant to its decision making. In our context, the fading channel gain $g_k$ is naturally considered as the type of user $k$'s, since its decision (in terms of power) can only rely on $g_k$.

Note that this is a *continuous game*[1] with discrete states, since each player's action $p_k$ can take any value satisfying the power constraint $p_k \in [0, P_k^{\max}]$ and the channel state $g_k$ is finite $g_k = g_+$ or $g_-$ for all $k$.

## IV. Bayesian equilibrium

### A. Definition of Bayesian equilibrium

What we can expect of the outcome from a Bayesian game if every selfish and rational (rational player means a player chooses the best response given its information) participant start to play the game? Generally speaking, the process of such players' behaviors usually results in Bayesian equilibrium, which represents a common solution concept for Bayesian games. In many cases, it represents a "stable" result of learning and evolution of all participants. Therefore, it is important to characterize such a equilibrium point, since it concerns the performance prediction of a distributed system.

Now, let $\{\hat{p}_k(\cdot), p_{-k}(\cdot)\}$ denote the strategy profile where all players play $p(\cdot)$ except player $k$ who plays $\hat{p}_k(\cdot)$, we can then describe player $k$'s payoff as:

$$u_k(\hat{p}_k, p_{-k}) = u_k(p_1, \dots, p_{k-1}, \hat{p}_k, p_{k+1}, \dots, p_K)$$

**Definition IV.1.** *(Bayesian equilibrium)*

*The strategy profile $p^\star(\cdot) = \{p_k^\star(\cdot)\}_{k\in\mathcal{K}}$ is a (pure strategy) Bayesian equilibrium, if for all $k \in \mathcal{K}$, and for all $p_k(\cdot) \in \mathcal{P}_k$ and $p_{-k}(\cdot) \in \mathcal{P}_{-k}$*

$$\bar{u}_k\left(p_k^\star, p_{-k}^\star\right) \geq \bar{u}_k\left(p_k, p_{-k}^\star\right)$$

*where we define $\bar{u}_k \triangleq \mathbb{E}_{\mathbf{g}}[u_k]$.*

From this definition, it is clear that at the Bayesian equilibrium no player can benefit by changing its strategy while the other players keep theirs unchanged. Note that in a strategic-form game with complete information each player chooses a concrete action, whereas in a Bayesian game each player $k$ faces the problem of choosing a set or collection of actions (power strategy $p_k(\cdot)$), one for each type (channel gain $g_k$) it may encounter. It is also worth to mention that

[1]A continuous game extends the notion of a discrete game (where players choose from a finite set of pure strategies), it allow players to choose a strategy from a continuous pure strategy set

the action set of each player is independent of the type set, i.e., the actions available to user $k$ is the same for every its type.

## B. Characterization of Bayesian equilibrium set

For such a Bayesian game, it is of primary importance in investigating the existence and uniqueness of Bayesian equilibrium, since equilibrium point does not necessarily exist in general [6]. Here, we directly give our main result:

**Theorem IV.2.** *There exists a unique Bayesian equilibrium in the $K$-user MAC game $\mathcal{G}_{MAC}$.*

*Proof:* It is easy to prove the existence part, since the strategy space $p_k$ is convex, compact and nonempty for each $k$; the payoff function $u_k$ is continuous in both $p_k$ and $p_{-k}$; and $u_k$ is concave in $p_k$ for any $p_{-k}$ [6].

In order to prove the uniqueness part, we should rely on a sufficient condition given in [18]: a non-cooperative game has a unique equilibrium, if the nonnegative weighted sum of the payoff functions is *diagonally strictly concave*. We firstly give the definition:

**Definition IV.3.** *(Diagonally strictly concave)*

*A weighted nonnegative sum function $f(\mathbf{x}, \mathbf{r}) = \sum_{i=1}^{n} r_i \varphi_i(\mathbf{x})$ is called diagonally strictly concave for any vector $\mathbf{x} \in \mathbb{R}^{n\times 1}$ and fixed vector $\mathbf{r} \in \mathbb{R}^{n\times 1}_{++}$, if for any two different vectors $\mathbf{x}^0, \mathbf{x}^1$, we have*

$$\Omega(\mathbf{x}^0, \mathbf{x}^1, \mathbf{r}) \triangleq (\mathbf{x}^1 - \mathbf{x}^0)^{\mathrm{T}} \delta(\mathbf{x}^0, \mathbf{r}) + (\mathbf{x}^0 - \mathbf{x}^1)^{\mathrm{T}} \delta(\mathbf{x}^1, \mathbf{r}) > 0 \tag{5}$$

*where $\delta(\mathbf{x}, \mathbf{r})$ is called pseudo-gradient of $f(\mathbf{x}, \mathbf{r})$, defined as*

$$\delta(\mathbf{x}, \mathbf{r}) \triangleq \begin{bmatrix} r_1 \frac{\partial \varphi_1}{\partial x_1} \\ \vdots \\ r_n \frac{\partial \varphi_n}{\partial x_n} \end{bmatrix}. \tag{6}$$

We start with the following lemma.

**Lemma IV.4.** *The weighted nonnegative sum of the average payoffs $\bar{u}_k$ in $\mathcal{G}_{MAC}$ is diagonally strictly concave.*

*Proof:* Write the weighted nonnegative sum of the average payoffs as:

$$f^u(\mathbf{p},\mathbf{r}) \triangleq \sum_{k=1}^{K} r_k \bar{u}_k(\mathbf{p}), \tag{7}$$

where $\mathbf{p} = [p_1 \ldots p_K]^{\mathrm{T}}$ is the transmit power vector, $\mathbf{r} = [r_1 \ldots r_K]^{\mathrm{T}}$ is a nonnegative vector assigning weights $r_1, \ldots, r_K$ to the average payoffs $\bar{u}_1, \ldots, \bar{u}_K$, respectively. Similar to (6), we let $\delta^u(\mathbf{p},\mathbf{r}) \triangleq [r_1 \frac{\partial \bar{u}_1}{\partial p_1} \ldots r_K \frac{\partial \bar{u}_K}{\partial p_K}]^{\mathrm{T}}$ be the pseudo-gradient of $f^u(\mathbf{p},\mathbf{r})$. Now, we define

$$p_k \triangleq p_k\left(g_-\right) \ \forall k,$$

the transmit power of player $k$ when its channel gain is $g_-$. Since we have shown from Lagrangian that, at the equilibrium, the power constraint is satisfied with equality, i.e., $\sum_{g_k} p_k\left(g_k\right) = P_k^{\max}$, we can write $P_k^{\max} - p_k = p_k\left(g_+\right) \ \forall k$, as the transmit power when its channel gain is $g_+$. Therefore, it is easy to find that the average payoff $\bar{u}_k$ can be actually transformed into a weighted sum-log function, as follows

$$\bar{u}_k(p_k) = \sum_i \omega_i \log\left[1 + \frac{\alpha_k^i + \beta_k^i p_k}{\sigma^2 + \sum_{j \neq k}\left(\alpha_j^i + \beta_j^i p_j\right)}\right]$$

where $i$ represents the index for different jointly probability events, $\omega_i$ represents the corresponding probability for index $i$. Note that the following conditions hold for all $i, k$

$$\alpha_k^i + \beta_k^i p_k \geq 0,\ \alpha_k^i > 0,\ \beta_k^i \neq 0,\ \sigma^2 > 0$$

Now, we can write the pseudo-gradient $\delta^u$ as

$$\delta^u(\mathbf{p},\mathbf{r}) = \begin{bmatrix} r_1 \frac{\partial u_1}{\partial p_1} \\ \vdots \\ r_K \frac{\partial u_K}{\partial p_K} \end{bmatrix} = \begin{bmatrix} r_1 \sum_i \beta_1^i \phi_i^{-1}(\mathbf{p}) \\ \vdots \\ r_K \sum_i \beta_K^i \phi_i^{-1}(\mathbf{p}) \end{bmatrix} = \sum_i \begin{bmatrix} r_1 \beta_1^i \phi_i^{-1}(\mathbf{p}) \\ \vdots \\ r_K \beta_K^i \phi_i^{-1}(\mathbf{p}) \end{bmatrix}$$

where function $\phi_i\left(\mathbf{x}\right)$ is defined as

$$\phi_i(\mathbf{x}) \triangleq \sigma^2 + \sum_{k=1}^{K}\left(\alpha_k^i + \beta_k^i x_k\right)$$

To check the diagonally strictly concave condition (5), we let $\mathbf{p}^0, \mathbf{p}^1$ be two different vectors satisfying the power constraint, and define

$$\begin{aligned}
\Omega^u(\mathbf{p}^0, \mathbf{p}^1, \mathbf{r}) &\triangleq (\mathbf{p}^1 - \mathbf{p}^0)^{\mathrm{T}} \delta^u(\mathbf{p}^0, \mathbf{r}) + (\mathbf{p}^0 - \mathbf{p}^1)^{\mathrm{T}} \delta^u(\mathbf{p}^1, \mathbf{r}) \\
&= (\mathbf{p}^1 - \mathbf{p}^0)^{\mathrm{T}} \left[\delta^u(\mathbf{p}^0, \mathbf{r}) - \delta^u(\mathbf{p}^1, \mathbf{r})\right] \\
&= [\Delta p_1 \ \cdots \ \Delta p_K] \begin{bmatrix} r_1 \sum_i \beta_1^i \left(\phi_i^{-1}(\mathbf{p}^0) - \phi_i^{-1}(\mathbf{p}^1)\right) \\ \vdots \\ r_K \sum_i \beta_K^i \left(\phi_i^{-1}(\mathbf{p}^0) - \phi_i^{-1}(\mathbf{p}^1)\right) \end{bmatrix} \\
&= \sum_i \left[\phi_i^{-1}(\mathbf{p}^0) - \phi_i^{-1}(\mathbf{p}^1)\right] \zeta_i \\
&= \sum_i \phi_i^{-1}(\mathbf{p^0}) \phi_i^{-1}(\mathbf{p^1}) \zeta_i^2
\end{aligned}$$

where $\Delta p_k$ and $\zeta_i$ are defined as

$$\begin{aligned}
\Delta p_k &\triangleq p_k^1 - p_k^0 \\
\zeta_i &\triangleq \sum_{k=1}^{K} r_k \beta_k^i \Delta p_k.
\end{aligned}$$

Since $\mathbf{p}^0, \mathbf{p}^1$ are assumed to be two different vectors, we must have $\Delta\mathbf{p} = [\Delta p_1 \cdots \Delta p_K]^{\mathrm{T}} \neq \mathbf{0}$. Now, we can draw a conclusion from the equation above: $\Omega^u(\mathbf{p}^0, \mathbf{p}^1, \mathbf{r}) > 0$. This is because: $(1)$ the first part $\phi_i^{-1}(\mathbf{p}^0)\phi_i^{-1}(\mathbf{p^1}) > 0$ for all $i$, since $\sigma^2 > 0$ and $\alpha_k^i + \beta_k^i p_k \geq 0$ for all $i, k$; $(2)$ the second part $\zeta_i^2 \geq 0$ for all $i$, and there exists at least one nonzero term $\zeta_i^2$, due to $\Delta\mathbf{p} \neq 0$ and $r_k \neq 0, \beta_k^i \neq 0$ for all $i, k$. Therefore, the summation of all the products of the first and the second terms must be positive. From Definition IV.3, the sum-payoff function $f^u(\mathbf{p}, \mathbf{r})$ satisfies the condition of diagonally strictly concave. This completes the proof of this lemma. ■

Since our sum-payoff function $f^u(\mathbf{p}, \mathbf{r})$ given in (7) is diagonally strictly concave, from the Theorem $2$ in [18], we have the uniqueness of Nash equilibrium in our game $\mathcal{G}_{MAC}$. ■

## V. Optimal symmetric strategies

Bayesian game-theoretic approach provides us a better understanding of the wireless resource competition existing in the fading MAC when every mobile device acts as a selfish and rational decision maker (this means a device always chooses the best response given its information). In such a non-cooperative game, there is no action restriction rule nor action priority policy to influence the decision making process of each player. In other words, mobile devices are entirely "free" to make their own choices.

However, from the global system performance perspective, it is usually inefficient to give complete “freedom” to mobile devices and let them take decisions without any policy control over the network. It is very interesting to note that a similar situation happens in the market economy, where consumers can be modeled as players to complete for the market resources. In the famous literature *The Wealth of Nations*, Adam Smith[2] expounded how rational self-interest and competition can lead to economic prosperity and well-being through macroeconomic adjustments. For example, all states today have some form of macroeconomic control over the market that removes the free and unrestricted direction of resources from consumers and prices such as tariffs and corporate subsidies.

Moreover, in introduced Bayesian game model some basic assumptions may not be satisfied in the current wireless systems. One of the crucial questions is related to the acceptance level of the distributed computational complexity, i.e., a modern mobile device may not be “intelligent” and “powerful” enough to act like a rational player (to find the best response within limited time under complex situations). In this case, it may be better to assign them the pre-analyzed strategies, guiding them how to react under all kinds of different situations. Thus, there is no surprise why the wireless networks are often designed in a manner such that all identical mobile devices follow the same policy and strategy (means that they react in the same way when they fact the same situation).

Typically, wireless service providers would like to design an appropriate policy to efficiently manage the wireless resource so that the global network performance can be optimized or enhanced to a certain theoretical limit. Based on the Bayesian game settings, we provide in this part a special discussion on the practical concerns of resource allocation design, i.e., the optimal symmetric resource allocation strategy. This result can be also treated as a theoretical upper-bound for the performance measurement of Bayesian equilibrium.

We now introduce an necessary assumption, as follows

**Assumption V.1.** *Mobile devices are designed to use the same power transmission strategies, i.e., they apply the same strategy if their observations on the channel states are symmetric. And we assume the mobile devices have the same average power constraint, i.e.,* $P_1^{\max} = P_2^{\max} \triangleq P^{\max}$.

[2] A Scottish moral philosopher, pioneer of political economy, father of modern economics.

### *A. Two channel states*

For simplicity of our presentation, We first consider the scenario of two channel states, i.e., $g_k$ can be either $g_-$ or $g_+$ with probability $\rho_-$ and $\rho_+$, respectively. In fact, the analysis of multi-user MAC can be extended in a similar way. According to Assumption V.1, we define

$$p_- \triangleq p_1(g_-) = p_2(g_-)$$
$$p_+ \triangleq p_1(g_+) = p_2(g_+)$$

and we have $\rho_- p_- + \rho_+ p_+ = P^{\max}$. Write user 1's average payoff as (WloG, we consider user 1 in the following context, since the problem is symmetric for user 2)

$$\begin{aligned} \bar{u}_1 &= \mathbb{E}_{g_1,g_2}\left[\log_2\left(1+\tfrac{g_1 p_1(g_1)}{\sigma^2+g_2 p_2(g_2)}\right)\right] \\ &= \rho_-^2 \log_2\left(1+\tfrac{g_- p_-}{\sigma^2+g_- p_-}\right) + \rho_-\rho_+ \log_2\left(1+\frac{g_- p_-}{\sigma^2+\frac{g_+(P^{\max}-\rho_- p_-)}{\rho_+}}\right) + \\ &\quad \rho_-\rho_+ \log_2\left(1+\frac{\frac{g_+(P^{\max}-\rho_- p_-)}{\rho_+}}{\sigma^2+g_- p_-}\right) + \rho_+^2 \log_2\left(1+\frac{\frac{g_+(P^{\max}-\rho_- p_-)}{\rho_+}}{\sigma^2+\frac{g_+(P^{\max}-\rho_- p_-)}{\rho_+}}\right) \end{aligned}$$

Now, $\bar{u}_1$ is transformed into a function of $p_-$, write it as $\bar{u}_1(p_-)$. To maximize the average achievable rate, user 1 needs to solve the following optimization problem, as mentioned in (2)

$$\begin{aligned} \max_{p_-} \quad & \bar{u}_1(p_-) \\ s.t. \quad & 0 \le p_- \le P^{\max} \end{aligned}$$

Under Assumption V.1, it can be shown that (due to the symmetric property) this single-user maximization problem is equivalent to the multi-user sum average rate maximization problem, i.e., $\max(\bar{u}_1+\bar{u}_2)$, which is our object in this section.

But unfortunately, $\bar{u}_1$ may not be a convex function [17], so the single-user problem may not be a convex optimization problem. It can be further verified that $\bar{u}_1$ is convex under some special conditions, depending on all the parameters $g_-, g_+, \rho_-, \rho_+, P^{\max}$ and $\sigma^2$. Here, we will not discuss all the convex cases, but only focus on the high SNR regime (means that the noise can be omitted compared to the signal strength). In this case, we have

$$\lim_{\sigma^2\to 0} u_1 = \rho_-\rho_+\left[\log_2\left(1+\frac{g_- p_-}{\frac{g_+(P^{\max}-\rho_- p_-)}{\rho_+}}\right) + \log_2\left(1+\frac{\frac{g_+(P^{\max}-\rho_- p_-)}{\rho_+}}{g_- p_-}\right)\right] + \rho_-^2 + \rho_+^2$$

This function is strict convex. To be more precise, it is decreasing on $\left[0, \frac{g_+ P^{\max}}{g_- \rho_+ + g_+ \rho_-}\right)$ and increasing on $\left(\frac{g_+ P^{\max}}{g_- \rho_+ + g_+ \rho_-}, \frac{P^{\max}}{\rho_-}\right]$, and the solution is given by

$$\{p_-^\star, p_+^\star\} = \begin{cases} \left\{0, \frac{P^{\max}}{\rho_+}\right\}, & \frac{g_+}{\rho_+} \geq \frac{g_-}{\rho_-} \\ \left\{\frac{P^{\max}}{\rho_-}, 0\right\}, & \frac{g_+}{\rho_+} < \frac{g_-}{\rho_-} \end{cases}$$

Note that the choice of the optimal symmetric strategy, in our settings, is to focus full power on a channel state. It depends not only on the channel conditions but also on the probability of the channel states. This result implies that, in the high SNR regime, the optimal symmetric power strategy is to transmit information in an "opportunistic" way [2]. It is also clear that an appropriate "trade-off" should be handled between choosing the channel condition $g$ (which is related to the transmission rate) and choosing the probability $\rho$ (which is related to the probability of transmission collision).

### B. *Multiple channel states*

In this subsection, we discuss the extension to arbitrary $L$ ($L > 2$) channel states.

**Assumption V.2.** *Each user's channel gain $g_k$ has $L$ positive states, which are $a_1, \ldots, a_L$ with probability $\rho_1, \ldots, \rho_L$ respectively (WloG, $a_1 < \ldots < a_L$), and we have $\sum_{\ell=1}^{L} \rho_\ell = 1$.*

Based on Assumption V.1, we define $p_\ell \triangleq p_1(g_\ell) = p_2(g_\ell), \ \ell = 1, \ldots, L$, the transmit power when a user's channel gain is $g_\ell$. As previously mentioned, in this part, our object is to maximize the sum average rate of the system, i.e., $\max \sum_k \bar{u}_k$. Under the symmetric assumption, it is equivalent to the following single-user maximization problem

$$\begin{aligned} \max_{\mathbf{p}} \quad & \sum_i \sum_j \rho_i \rho_j \log_2\left(1 + \frac{g_i p_i}{\sigma^2 + g_j p_j}\right) \\ \text{s.t.} \quad & \sum_i \rho_i p_i = P^{\max} \\ & p_i \geq 0, \ i = 1, \ldots, L \end{aligned} \tag{8}$$

where $\mathbf{p}$ is now defined as $\mathbf{p} = \{p_1, \ldots, p_L\}$. This optimization problem is difficult, since the objective function is again nonconvex in $\mathbf{p}$. However, we can consider a relaxation of the optimization by introducing a lower bound [19]

$$\alpha \log z + \beta \leq \log(1 + z) \tag{9}$$

where $\alpha$ and $\beta$ are chosen specified as below

$$\begin{cases} \alpha = \frac{z_0}{1+z_0} \\ \beta = \log(1+z_0) - \frac{z_0}{1+z_0}\log z_0 \end{cases} \tag{10}$$

we say the lower bound (9) is tight with equality at a chosen value $z_0$.

Let us consider the lower bound (denoted as $\xi$) by using (9) to the objective function expressed in (8)

$$\xi(\mathbf{p}) \triangleq \sum_i \sum_j \rho_i \rho_j \left[ \alpha_{i,j} \log_2 \left( \frac{g_i p_i}{\sigma^2 + g_j p_j} \right) + \beta_{i,j} \right] \tag{11}$$

which is still nonconvex, and so it is not concave in $\mathbf{p}$. However, by using Geometric Programming [20] we can transform (11) into a convex optimization problem. Define a mapping operation $\tilde{x}$ as

$$\tilde{x} \triangleq \log_2 x$$

Then (11) can be rewritten as

$$\xi(\tilde{\mathbf{p}}) = \sum_i \sum_j \rho_i \rho_j \alpha_{i,j} (\tilde{g}_i + \tilde{p}_i) - \sum_i \sum_j \rho_i \rho_j \alpha_{i,j} \log_2 \left( \sigma^2 + 2^{(\tilde{g}_j + \tilde{p}_j)} \right) + \sum_i \sum_j \rho_i \rho_j \beta_{i,j} \tag{12}$$

Now, it is easy to verify that the lower bound $\xi$ is concave in the transformed set $\tilde{\mathbf{p}}$, since the log-sum-exp function is convex. The corresponding constraint set is convex. So, the Karush-Kuhn-Tucker (KKT) condition of the optimization is sufficient and necessary for the optimality. To derive the KKT conditions, form the Lagrangian (denoted by $\mathcal{L}$):

$$\begin{aligned} \mathcal{L}(\tilde{\mathbf{p}}, \nu, \lambda) = & \sum_i \sum_j \rho_i \rho_j \alpha_{i,j} (\tilde{a}_i + \tilde{p}(a_i)) - \sum_i \sum_j \rho_i \rho_j \alpha_{i,j} \log_2 \left( \sigma^2 + 2^{(\tilde{a}_j + \tilde{p}(a_j))} \right) + \\ & + \sum_i \sum_j \rho_i \rho_j \beta_{i,j} - \nu \left( \sum_i 2^{(\tilde{a}_i + \tilde{p}(a_i))} - \bar{P} \right) + \sum_i \lambda_i 2^{(\tilde{a}_i + \tilde{p}(a_i))} \end{aligned} \tag{13}$$

the KKT conditions are

$$\rho_\ell \sum_j \rho_j \alpha_{\ell,j} - \rho_\ell \left( \frac{2^{(\tilde{a}_\ell + \tilde{p}(a_\ell))}}{\sigma^2 + 2^{(\tilde{a}_\ell + \tilde{p}(a_\ell))}} \right) \sum_i \rho_i \alpha_{i,\ell} + (\ln 2)(\lambda_\ell - \nu) 2^{(\tilde{a}_\ell + \tilde{p}(a_\ell))} = 0, \ \forall \ell$$

$$\lambda_\ell 2^{(\tilde{a}_\ell + \tilde{p}(a_\ell))} = 0, \ \forall \ell$$

and $\nu, \lambda_\ell \geq 0, \ \forall \ell$, where $\nu$ and $\lambda_\ell$ are dual variables associated with the power constraints and positive constraint, respectively. From the second equation above, one can find $\lambda_\ell = 0, \ \ell = 1, \ldots, L$, since $2^{(\tilde{a}_\ell + \tilde{p}(a_\ell))}$ is a nonzero term. Define $x_\ell \triangleq 2^{(\tilde{a}_\ell + \tilde{p}(a_\ell))}, \ \ell = 1, \ldots, L$, the equivalent KKT conditions can be simply written as a quadratic equation

$$A_\ell x_\ell^2 + B_\ell x_\ell + C_\ell = 0, \quad \forall \ell$$

where the parameters are

$$\begin{cases} A_\ell &= \nu \ln 2, \quad \forall \ell \\ B_\ell &= \rho_\ell \sum_i \rho_i \left(\alpha_{i,\ell} - \alpha_{\ell,i}\right) + \sigma^2 \nu \ln 2, \quad \forall \ell \\ C_\ell &= -\rho_\ell \sigma^2 \sum_i \rho_i \alpha_{\ell,i}, \quad \forall \ell \end{cases} \tag{14}$$

and $\nu \geq 0$. Note that $A_\ell$ and $B_\ell$ are functions of $\nu$, we can write them as $A_\ell(\nu)$ and $B_\ell(\nu)$. Since $x_\ell \geq 0$, the solution to the KKT conditions can only be one of the roots to the quadratic equation, i.e.,

$$p_\ell^\star = \frac{-B_\ell(\nu) + \sqrt{B_\ell^2(\nu) - 4A_\ell(\nu)C_\ell}}{2a_\ell A_\ell(\nu)}, \quad \forall \ell \tag{15}$$

where $\nu$ is chosen such that $\sum_\ell \rho_\ell p_\ell^\star = P^{\max}$. Thus, for some fixed value of $\alpha, \beta$, we can maximize the lower-bound $\xi$ (11) by directly applying the result from (15). Then, it is natural to improve the bound periodically. Bases on all presented above, we propose a Lower Bound Tightening (LBT) algorithm.

**Algorithm 1** Lower Bound Tightening (LBT)

**Initialize** $t = 0;\ \nu = 0;\ \alpha_{i,j}^{(t)} = 1$, for $i = 1, \ldots, L,\ j = 1, \ldots, L.$

**repeat**

  **repeat**

    $\nu = \nu + \Delta\nu$

    **for** $i = 1$ to $L$ **do**

      update $A_i, B_i, C_i$ using (14)

      $p^\star(a_i) = \frac{-B_i + \sqrt{B_i^2 - 4A_iC_i}}{2a_iA_i}$

    **end for**

  **until** $\sum_i \rho_i p^\star(a_i) = P^{\max}$

  **for** $i = 1$ to $L$ and $j = 1$ to $L$ **do**

    $z_{i,j}^{(t)} = \frac{a_i p^\star(a_i)}{\sigma^2 + a_j p^\star(a_j)};\ \alpha_{i,j}^{(t+1)} = \frac{z_{i,j}^{(t)}}{1 + z_{i,j}^{(t)}}.$

  **end for**

  $t = t + 1$

**until** converge

The algorithm convergence can be easily proved, since the objective is monotonically increasing at each iteration. However, the global optimum is not always guaranteed, due to the nonconvex property.

## VI. Numerical Results

In this section, numerical results are presented to validate our theoretical claims. For Fig. 1 and Fig. 2, the network parameters are chosen as $\rho_- = \rho_+ = 0.5, P^{\max} = 1$ and $\sigma^2 = 0.1$.

First, we show the existence and uniqueness of Bayesian equilibrium in the scenario of two-user fading MAC. In Fig. 1, on the left, we assume the channel gains are $g_- = 1, g_+ = 3$; on the right, we assume $g_- = 1, g_+ = 10$. In both $X$ and $Y$ axis, the $p_1$ and $p_2$ represent the power allocated by user $1$ and user $2$ when the channel gain is $g_-$. The curves $r_1(p_2)$ and $r_2(p_1)$ represent the best-response functions of user $1$ and user $2$, respectively. As expected, the Bayesian equilibrium is unique in both cases, i.e., (0.6,0.6) and (0.5,0.5).

Second, we investigate the efficiency of Bayesian equilibrium from the viewpoint of global average network performance. The $X$ axis, SNR is defined as the ratio between the power constraint $P^{\max}$ and the noise variance $\sigma^2$. In Fig. 2, again, on the left, we assume $g_- = 1, g_+ = 3$; on the right, we assume $g_- = 1, g_+ = 10$. The curve "Pareto" represents the Nash equilibrium in the waterfilling game, in which users have complete information. This gives the upper bound for our Bayesian equilibrium, since it is also the Pareto optimal solution [7]. The curve "Uniform" represents the time-domain uniform power allocation. Since this is the strategy when users have no information about the channel gains, obviously it corresponds to the lower bound. The curve "Symmetric" represents the optimal symmetric strategy we presented in section V, this can be treated as a weaker upper bound (inferior to the Pareto optimality) for the Bayesian equilibrium. From the slopes of these curves, we can clearly observe the inefficiency of Bayesian equilibrium, especially in the high SNR regime. We can conclude it as follows: in our game $\mathcal{G}_{MAC}$, users with incomplete information does improve the global network performance (comparing to the scenario in which users has no information), however, it does not improve the performance slope.

Finally, we show the convergence behavior of the lower bound tightening (LBT) algorithm. In Fig. 3, we choose the parameters as $L = 3, g_1 = 1, g_2 = 2, g_3 = 3, \rho_1 = \rho_2 = \rho_3 = 1/3$. The sum capacity versus the SNR are plotted for five iterations, and the upper bound is achieved by exhaustive search. As expected, one can easily observe the convergence behavior. In the low

SNR regime, we can find that the algorithm converges to the local maximum instead of global maximum. However, we also find that the performance of the local optimum is improved while the SNR is increasing.

## VII. Conclusion

We presented a Bayesian game-theoretic framework for distributed resource allocation in fading MAC, where users are assumed to have only local information about the network channel gain states. By introducing the assumption of finite channel states, we successfully found a analytical way to characterize the Bayesian equilibrium set. First, we proved the existence and uniqueness. Second, the inefficiency was shown from numerical results. Furthermore, we analyzed the optimal symmetric power strategy based on the practical concerns of resource allocation design. Future extension is considered to improve the efficiency of Bayesian equilibrium through pricing or cooperative game-theoretic approaches.

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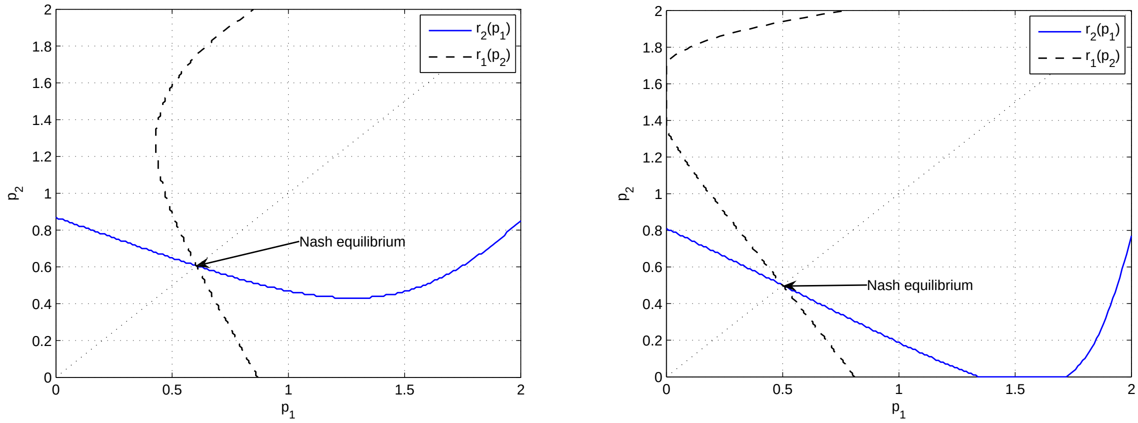


Fig. 1. The uniqueness of Bqyesian equilibrium (left: $g_- = 1, g_+ = 3$, right:$g_- = 1, g_+ = 10$).

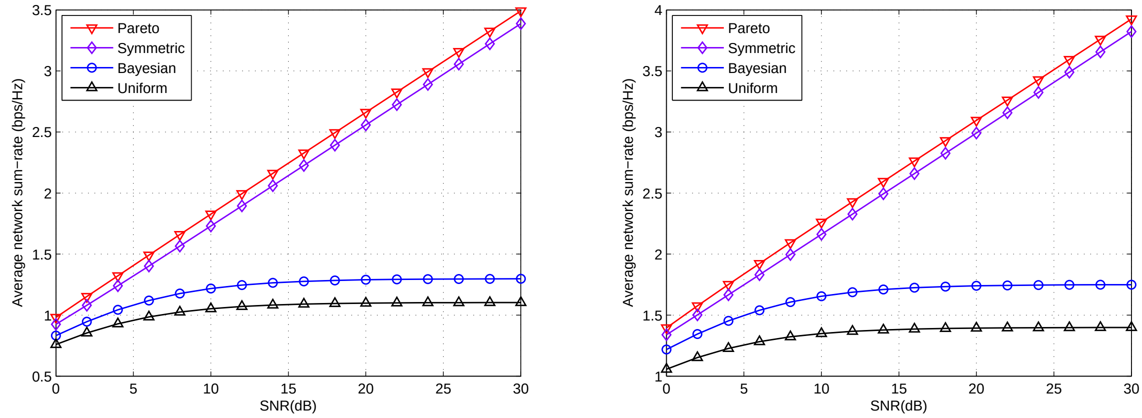


Fig. 2. Average network sum-rate (left: $g_- = 1, g_+ = 3$, right: $g_- = 1, g_+ = 10$).

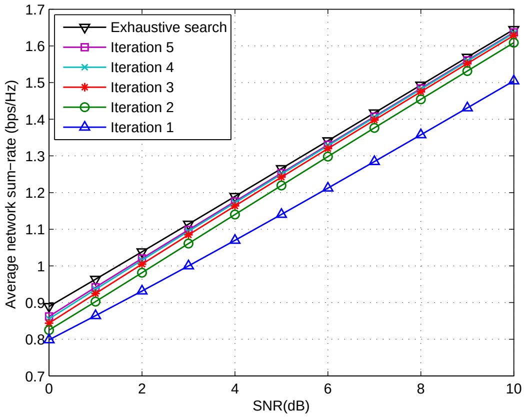


Fig. 3. The convergence of the lower bound tightening (LBT) algorithm.